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# **A Gravitational Theory of Quantum Mechanics**

by

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**Thesis**

**Submitted to the University of Warwick**

**for the degree of**

**Doctor of Philosophy**

**Department of Physics**

**December 1996**

Page  
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original  
thesis

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# Acknowledgments

I have benefited enormously from the friendly and supportive atmosphere in both the Physics department and the Mathematics Institute at Warwick. I am particularly grateful for the helpful comments, constructive criticism and academic freedom given by my supervisor Dr Hyland. The interest and encouragement from Professor Rowlands has also been heartening. At times I have needed help and advice from outside the University of Warwick; of all the people who I have turned to, Professor Isham stands out, not just for his knowledge and expertise, but also for his generous advice and friendly manner.

These studies started with an Interdisciplinary Mathematics MSc/PhD: I am indebted to Dr Micallef for helping me to enrol as his student, and to my fellow students Tammo Diemer and Stefan Halverscheid for invaluable help with the mathematics courses.

My work was funded by a Warwick Graduate Award in my first year and by the Physics department itself in subsequent years. I am most grateful for this financial support.

Karen and I have two children, Simon and Zoe. My academic work was only possible because Karen worked to support the whole family while I studied. Many thanks are also due to the large group of friends and family (particularly my mother) who have helped our family during this demanding period.

# Declarations

This thesis is my own work. Chapter 2 is preparatory - it presents known results about quantum mechanics as required for the innovative work which follows. Chapter 3 describes a new classical model for quantum logic and gives an original, informal, derivation of the Hilbert space structure. Chapters 4 and 5 present selected results from general relativity with particular emphasis on closed timelike curves and geons - the results are well-known, but the presentation here is original. Chapter 6 describes an original explanation for quantum mechanics. Chapter 7 uses the explanation given in chapter 6 to formally construct an example of an orthomodular lattice (characteristic of quantum mechanics) from propositions about structures in classical general relativity - this has never been done before. Subsequent chapters consider the further implications of the work described in chapters 6 and 7. Chapter 8 is based on the paper by Friedman and Sorkin, but it extends, for the first time, their work from quantum gravity to classical general relativity and provides a justification for their assumptions.

An outline of this work was presented at the *5th UK Conference on Conceptual and Philosophical Problems in Physics* held at Oxford in September 1996 (a report of which was submitted to the eprint archive quant-ph/9609021). The ideas and conclusions, as described in chapters 6 and 7, have been accepted for publication in *Foundations of Physics Letters* (to appear in February 1997).

# Abstract

An explanation for quantum mechanics is given in terms of a classical theory (general relativity) for the first time. Specifically, it is shown that certain structures in classical general relativity can give rise to the non-classical logic normally associated with quantum mechanics.

An artificial classical model of quantum logic is constructed to show how the Hilbert space structure of quantum mechanics is a natural way to describe a measurement-dependent stochastic process.

A 4-geon model of an elementary particle is proposed which is asymptotically flat, particle-like and has a non-trivial causal structure. The usual Cauchy data are no longer sufficient to determine a unique evolution; the measurement apparatus itself can impose further non-redundant boundary conditions. When measurements of an object provide additional non-redundant boundary conditions, the associated propositions would fail to satisfy the distributive law of classical physics.

Using the 4-geon model, an orthomodular lattice of propositions, characteristic of quantum mechanics, is formally constructed within the framework of classical general relativity.

The model described provides a classical gravitational basis for quantum mechanics, obviating the need for quantum gravity. The equations of quantum mechanics are unmodified, but quantum behaviour is not universal; classical particles and waves could exist and there is no graviton.



# Chapter 1

## Introduction

Quantum mechanics is correct to all intents and purposes; it describes the microscopic world to a phenomenal degree of accuracy and has never been in conflict with experiment. Even the counter-intuitive predictions have been confirmed.

Any unification of quantum mechanics and general relativity must reproduce the whole of quantum mechanics as we know it. Attempts at unification based on a case by case explanation of experiments [35] can never be satisfactory and can never be complete since the range of possible experiments is unlimited.

Chapters 2 and 3 explore what is required to reproduce the whole of quantum mechanics. The foundations of quantum mechanics are investigated; Schrödinger's equation is shown to be a consequence of using a complex Hilbert space and Galilean symmetry operations - similarly the Klein-Gordon and Dirac equations follow from Poincaré symmetry. Next the use of a complex space is justified. Measurement-dependent stochastic processes are introduced with simple, but contrived, classical examples; heuristic arguments are used to justify the use of vector spaces to represent states, and of Hermitian operators to represent observables. Finally, proposition lattices are introduced; it is at this level that the distinction between quantum and classical mechanics is expressed completely and simply by the failure of Boolean logic. Quantum mechanics on a complex Hilbert space can be reproduced (probably

uniquely[7]) from non-Boolean proposition lattices.

The distinctive features of quantum mechanics can be summarised in a physical way by describing the evolution and measurement of a state as ‘*a measurement-dependent stochastic process*’. More precisely, the mathematical conclusion is that quantum mechanics is an orthomodular (non-Boolean) lattice of propositions.

By way of preparation, some results about unitary transformations are presented. Although they are well known, the main texts use expressions which assume a complex Hilbert space thus obstructing the application of their results to real vector spaces. Section 2.3 uses notation which is applicable to any field (including complex, real and quaternion cases). The desirability of using a complex vector space is one of the conclusions of this work.

Chapter 4 discusses some interesting features of general relativity which are relevant to quantum mechanics. General relativity offers a great richness that has hardly been utilised to date. An important local feature is the non-linearity of the theory. Although all testable predictions require only a linear version of the theory, the theory is inherently non-linear. One reason for the great richness of Einstein’s theory, which has not been fully exploited, is that the equations are inherently local, and although related to the global structure of spacetime, they neither prescribe nor are prescribed by the topology. New phenomena can be introduced by postulating non-trivial topological structures at either the microscopic or macroscopic level.

Geons (topological structures of spacetime) are presented as models for elementary particles in chapter 5. A key original feature of this work is the proposed novel class of spacetime structures with the potential to reproduce effects conventionally described as quantum mechanical. It must be stressed that although an explicit metric and topology with the requisite novel structure is not known, there is no known reason for them to be incompatible with general relativity.

The logical route from general relativity to quantum mechanics is shown schematically in figure 1.1.

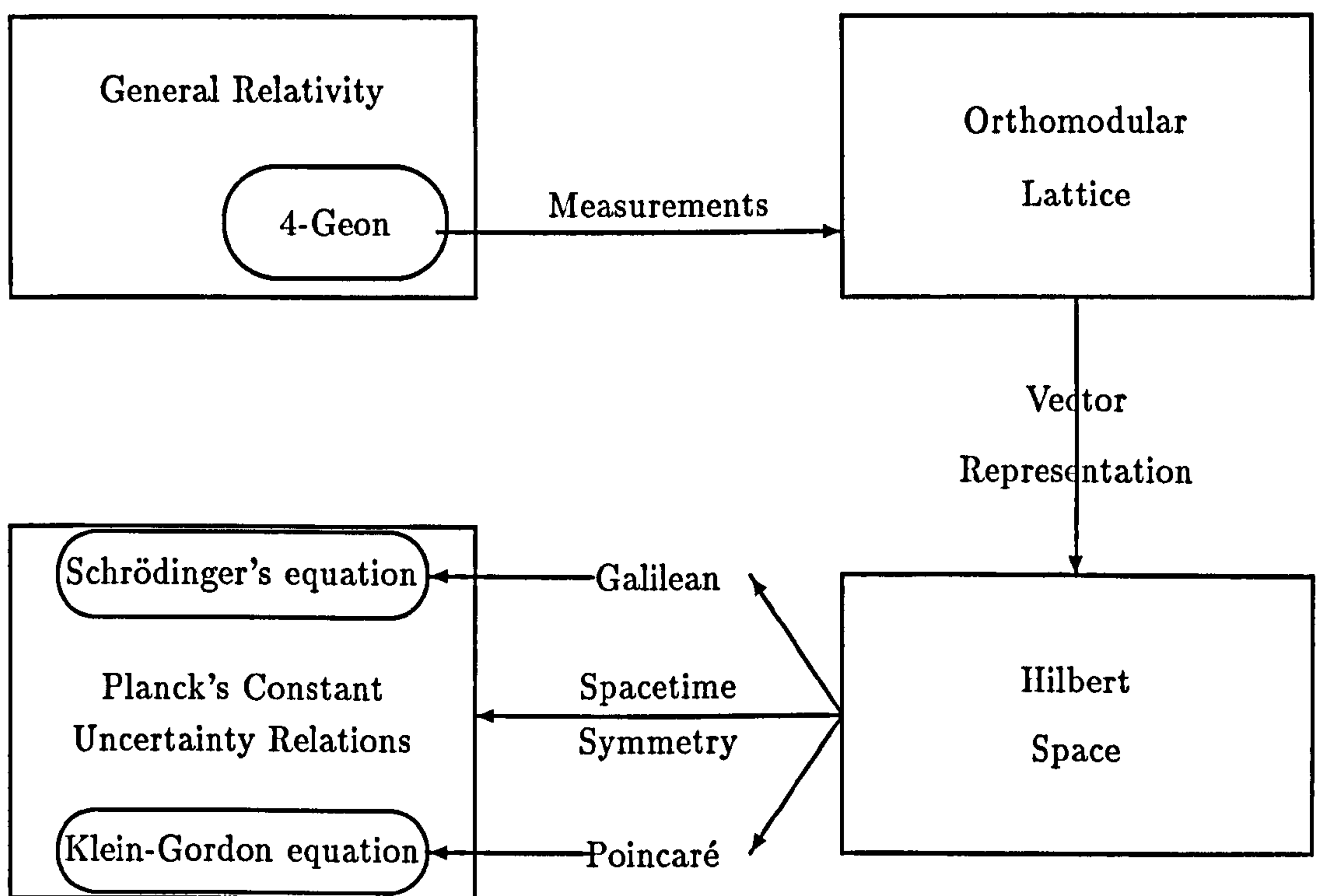


Figure 1.1: The route from general relativity to quantum mechanics.



The equations of quantum mechanics (Schrödinger's equation the Dirac equation *etc.*) are not derived directly from Einstein's equations; instead, it is shown how the logical structure characteristic of quantum mechanics can arise when measuring certain structures in general relativity, the logical structure can be represented by projections of a complex Hilbert space, and the familiar equations follow from the symmetries of space and time.

Although it may seem complicated, the majority of the analysis is well established. The route from projections of a complex Hilbert space representation to the familiar equations of Schrödinger, Dirac and Klein-Gordon is covered well by Ballentine [5] and Weinberg [40] the latter dealing with the relativistic case more fully, while Ballentine gives more detail but restricts attention to the non-relativistic case. The assumed symmetries of spacetime and the internal symmetries of the particle determine which equation is derived. In all cases a constant of proportionality is introduced - it is Planck's constant and cannot be zero. The commutation relations for position, momentum, angular momentum *etc.* are inescapable given the Hilbert space representation and the assumed symmetries.

The crucial and non-classical assumption is that a complex Hilbert space representation is required. However, it is known that certain classes of logic are isomorphic to projections of a complex Hilbert space; Beltrametti and Cassinelli [7] describe how this is established. The process they describe, limits itself to a Hilbert space over the real, complex or quaternion numbers. We will see that the real case is trivial. The identification of observables with Hermitian operators occurs as part of this process.

We are left with a simple question of what logic our physical system follows. The analysis of Beltrametti requires that the propositions (yes/no questions that can be asked) fail to be distributive, but satisfy instead the weaker ortho-modularity condition. Until now it has been assumed that all classical systems (including those described by general relativity) have distributive propositions, whilst quantum me-

chanics, *for no known reason*, only has orthomodular proposition systems. We will show that not only can general relativity exhibit non-distributive propositions, but that these are the orthomodular propositions of quantum mechanics. In doing so a gravitationally based explanation for the logic of quantum mechanics is given for the first time.

## Chapter 2

# Foundations of Quantum Mechanics

The objective of this chapter is to reveal the essence of quantum mechanics, by showing the route from a Hilbert space to Schrödinger's equation *etc.*; this follows Ballentine's book [5] and uses Weinberg's book [40] for some of the relativistic equations. Both sources are adapted to cover real and quaternion Hilbert spaces. The remainder of the chapter introduces propositional calculus and describes orthomodular lattices which have the same logical structure as projections of a Hilbert space. It is in the logic of propositions that the fundamental difference between classical and quantum physics is most clearly seen. Furthermore, the Hilbert space structure is known to be not just a representation of the logic, but a *unique* vector space representation.

### 2.1 Unitary Operations

All observables in quantum mechanics are represented by a Hermitian operator. For any pure state,  $\Psi$ , the expected value,  $p$ , of the observable represented by the

Hermitian operator  $P$  is:

$$\langle p \rangle = (\Psi, P.\Psi) \quad (2.1)$$

where  $(\Psi, \Phi)$  denotes the inner product of  $\Psi$  and  $\Phi$ . Since any measurable result is an observable, the inner product and its properties play a central role in the formulation of quantum mechanics. Any linear transformation,  $\Psi \rightarrow \Psi' = U\Psi$  which preserves  $(\Psi, \Psi)$  is a unitary transformation and also conserves  $(\Psi, \Phi)$ . A unitary operator satisfies  $UU^\dagger = I$  see reference [40]. There also exist antiunitary operations which preserve the inner product but satisfy:

$$A.(c\Psi) = c^* A.\Psi \quad (2.2)$$

where  $c$  is a complex number. However, the product of two antilinear operators is a linear operator - a fact which as we shall see, makes them inappropriate for our purposes.

## 2.2 Ray Representations

Quantum mechanics does not simply relate physical states to elements of a vector space by a 1-1 mapping. The mapping is necessarily *many-to-one*, as can be seen by considering any physically measurable result as given by equation 2.1. A change from  $\Psi \rightarrow \Psi' = e^{i\phi}\Psi$  leaves the inner product unchanged, and hence any observable is independent of the phase,  $\phi$ . These equivalent vectors must all represent the same physical state because they are physically indistinguishable - any measurement must give the same result. Only an overall phase factor maps vectors to equivalent ones - if a different phase factor is applied to each component of a vector, then the result will be a new, physically distinct state giving different measurable results. This multivalued representation is called a ray representation, and has important consequences for the transformation of vectors under operations which are symmetries of spacetime. For a real vector field the only acceptable change is from  $\Psi \rightarrow -\Psi$  (corresponding to  $\phi = \pi$ ).



An important difference arises between real and complex valued vector spaces, because the ‘ray’ is continuous in the latter case. It is possible to move continuously from one complex vector to an equivalent one while always describing the same physical state. By contrast the two elements of a real ray,  $\Psi$  and  $-\Psi$ , are discrete.

Vector spaces which have more dimensions (degrees of freedom) than are required to describe measurable results can exhibit a similar (phase freedom) effect through rotations in the space of superfluous dimensions. This mechanism allows a real vector space endowed with additional degrees of freedom to have continuous rays. The simplest example is a two dimensional real vector space with the usual inner product; the matrix operator:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.3)$$

is equivalent to  $i$ , the square root of  $-1$ . A general rotation in this space leaves the inner product of any two vectors unchanged. A 2-D real space with an operator,  $J$ , is isomorphic to a 1-D complex space.

## 2.3 Symmetry Operations and Infinitesimal Generators

A continuous symmetry operation,  $R(s)$ , of spacetime is associated with a unitary operation,  $U(R(s))$  of the vectors of a Hilbert space; this leaves all physical quantities unchanged and hence preserves the symmetry. Antilinear operators,  $A(R(s))$  are ruled out since they could always be expressed as the product of two other operations *eg.*  $R(s) = R(s/2)R(s/2) \Rightarrow A(R(s)) = (A(R(s/2))A(R(s/2)))$ , which is unitary if  $A(R(s/2))$  is antiunitary<sup>1</sup>.

All continuous unitary operators,  $U(s)$  can be written in terms of infinitesimal generators such that:

$$U(s) = e^{sK} \quad (2.4)$$

---

<sup>1</sup>Strictly speaking the equality only holds up to a constant phase factor because we are working with a ray representation. However, the argument given still applies

where  $K$  is an anti-Hermitian operator.

In reference texts on the subject the generators are commonly denoted by

$$U(s) = e^{isK} \quad (2.5)$$

so that  $K$  is now Hermitian. I will not use this definition since it is restricted to complex vector spaces, whilst the procedure and results are valid for real and quaternion vector spaces.

It follows that  $K^2$  is Hermitian, that the eigenvalues of  $K$  are purely imaginary (hence zero or non-existent for a real vector space) and that the eigenvalues of  $K^2$  are negative. On a complex or quaternion vector space a Hermitian operator  $\pm iK$  (which is a scalar multiple of  $K$ ) can always be constructed. On a real vector space we would need to postulate an anti-Hermitian operator  $J$ , such that  $J^\dagger \equiv J^T = -J$ , and  $J$  commutes with  $K$ . On real vector spaces of dimension greater than one the operator  $J$ , defined in equation 2.3 would suffice. There is no such operator acting on real numbers or on real-valued functions,  $f(x)$ . A real vector space of dimension two with the operator  $J$  is isomorphic to a complex vector space.

## Equations of Motion

The equations of motion follow directly from the definition of  $H$  as the generator of time translations. For a vector  $\Psi$ , expressed in terms of a fixed basis, the change with time is given by:

$$\Psi(t + t_0) = e^{-tH} \Psi(t_0) \quad (2.6)$$

$$\frac{\partial \Psi(t)}{\partial t} = -H \Psi(t), \quad (2.7)$$

provided only that  $H$  does not contain an explicit time dependence (see [5]). Equation 2.7 implies:

$$\frac{\partial^2}{\partial t^2} \Psi = H^2 \Psi \quad (2.8)$$

which has the advantage of incorporating the Hermitian operator  $H^2$ .

The symmetries of spacetime actually determine the structure of the associated quantum mechanical wave equations - such as Schrödinger's equation and the Klein-Gordon equation. If a state is represented by an element of a vector space and observables are represented by operators as in expression 2.1, then the symmetry operations on space and time lead inevitably to the equations of motion and the relation between energy and momentum. There are significant differences between the non-relativistic case with Galilean symmetry and the relativistic case which has Poincaré symmetry: these are dealt with in turn.

## 2.4 Galilean Transformations

Galilean transformations are the normal symmetry operations of space rotations and translations together with time translations and the effect of a boost. A boost relates one inertial frame to another one moving at a constant relative velocity,  $v$ . The boost is defined by the equations:

$$x \rightarrow x' = x + vt \quad (2.9)$$

$$t \rightarrow t' = t \quad (2.10)$$

These symmetry operations describe non-relativistic ( $1 + 1$  dimensional) spacetime.

### 2.4.1 Commutation Relations

The symmetries of spacetime for non-relativistic particles are given by the Galilean group. Any combination of group operations will result in another operation because it is a group. All the symmetry operations acting on space vectors preserve their length; they are therefore unitary operations and can be expressed in terms of infinitesimal generators 2.4, which we denote by:  $P_i$  for space translations;  $J_i$  for space rotations;  $G_i$  for Galilean boosts and  $H$  for time translations. Groups can



be characterised by the commutation relations of the generators; for the Galilean group these are:

$$[P_i, P_j] = 0 \quad (2.11)$$

$$[P_i, H] = 0 \quad (2.12)$$

$$[J_i, P_j] = -\varepsilon_{ijk} P_k \quad (2.13)$$

$$[G_i, P_j] = 0 \quad (2.14)$$

$$[J_i, H] = 0 \quad (2.15)$$

$$[J_i, J_j] = -\varepsilon_{ijk} J_k \quad (2.16)$$

$$[J_i, G_j] = -\varepsilon_{ijk} G_k \quad (2.17)$$

$$[G_i, H] = -P_i \quad (2.18)$$

$$[G_i, G_j] = 0 \quad (2.19)$$

It should be remembered that the generators used here are the anti-Hermitian operators defined in equation 2.4 rather than the Hermitian forms (defined as in 2.5) commonly seen in the literature.

#### 2.4.2 Commutation Relations for Ray Representations

Corresponding to each symmetry operation on spacetime there must be a symmetry operation on the vector space of state vectors which preserves the inner product (and hence any observable); furthermore these symmetry operations must satisfy the same commutation relations - but only when considered as operations on the rays, because it is rays, rather than vectors, which represent physical states. Ray representations in a complex vector space allow an extra multiple of the identity in each of the commutation relations, because that introduces a physically insignificant phase factor as described above. The multiple must be a pure imaginary number in order for the commutator to be anti-Hermitian. Consequently, a real vector space must satisfy the commutation relations 2.19 exactly. To obtain the commutation

relations for a complex Hilbert space we need to evaluate the unknown imaginary multiple of the identity. Using the relation  $[[A,B],C] = [[C,B],A] + [[A,C],B]$  and requiring consistency of the unknown multiple gives:

$$[P_i, P_j] = 0 \quad (2.20)$$

$$[P_i, H] = 0 \quad (2.21)$$

$$[J_i, H] = 0 \quad (2.22)$$

$$[G_i, G_j] = 0 \quad (2.23)$$

$$[G_i, H] = -P_i \quad (2.24)$$

Undetermined multiples of the identity occur for commutation relations with  $J_i$  but these can all be removed by redefining  $J_i$ :

$$[J_i, P_j] = -\varepsilon_{ijk} P_k \quad (2.25)$$

$$[J_i, J_j] = -\varepsilon_{ijk} J_k \quad (2.26)$$

$$[J_i, G_j] = -\varepsilon_{ijk} G_k \quad (2.27)$$

This is identical to the relations given in equations 2.11 to 2.19 except for equation 2.14. There remains one unknown multiple of the identity, *ie.* 2.14 is replaced by:

$$[G_i, P_j] = i\delta_{ij}\lambda \quad (2.28)$$

The last relation is crucial for non-relativistic quantum mechanics; it can only have meaning for a complex vector space where an anti-Hermitian operator (which is also a non-zero multiple of the identity) can be defined. The real number  $\lambda$  turns out to be proportional to the mass of the particle.

There is an important relation between the operators which requires no further assumptions:

$$H = i\frac{P^2}{2\lambda} + H_0, \quad (2.29)$$

where  $\lambda$  is the real constant appearing in equation 2.28 and  $H_0$  is an operator which commutes with all the others and is either a purely imaginary multiple of the identity

or possibly the generator of an unrelated internal symmetry operation. Equation 2.29 can be confirmed by showing that  $H - iP^2/2\lambda$  commutes with all the generators of the Galilei group. Given the classical relation between time translation symmetry and energy conservation; and between space translations and momentum conservation it is natural to consider  $iH$  to be proportional to energy,  $iP$  to momentum and  $\lambda$  to mass - all with the *same* constant of proportionality. For a real vector space equation 2.29 must have  $\lambda = 0$ , giving the trivial case  $P^2 = 0$ . Equation 2.7 and 2.29 can be combined to give:

$$\frac{\partial}{\partial t}\Psi = \frac{i}{2\lambda}\mathbf{P}.\mathbf{P}\Psi + H_0\Psi \quad (2.30)$$

This is Schrödinger's equation, apart from the constant of proportionality,  $\hbar$  on the LHS. Note however, that the units of equation 2.30 are not those of Schrödinger's equation. From equation 2.4 the units of  $\mathbf{P}$  are  $L^{-1}$ ;  $H_0$  are  $T^{-1}$  and  $\lambda$  are  $L^{-3}T$  (which can be seen from equation 2.28). Planck's constant and the units of equation 2.30 will be discussed further in section 2.5.

## 2.5 Planck's Constant

To many people Planck's constant epitomises quantum mechanics. It certainly sets the scale for quantum effects to be significant. Any *derivation* of quantum mechanics is *expected* to give a value to Planck's constant. The purpose of this chapter is to judge these expectations and relate them to the gravitational theory of quantum mechanics being presented here. We shall see where Planck's constant arises and to what extent our theory, or any other theory, can predict its value.

Recalling from chapter 1 the route from general relativity to quantum mechanics. In chapter 6 we start by constructing non- distributive propositions, formally filling in the details by constructing an orthomodular lattice. From then on standard results from the literature are invoked; orthomodular lattices can be represented by projection operators on Hilbert spaces as described by Beltrametti[7].



Given the use of a Hilbert space and projection operators, the derivation of the familiar equations of quantum mechanics (Schrödinger's equation, the Dirac equation and the Klein-Gordon equation) and the form of the operators for momentum, position, angular momentum, spin and energy is described in Ballentine's book[5]. It is in the latter process that Planck's constant arises.

The appearance of the constant is shown most clearly in the case of Galilean symmetry (giving Schrödinger's equation), because here there exist unambiguous position operators which give a quick and clear construction for the velocity, and hence momentum, operator.

As in section 2.3 we denote the generators of time, space and angular displacements by  $H$ ,  $P$  and  $J$  respectively. They are anti-Hermitian operators as explained in chapter 2.3. The relation  $H = iP^2/2\lambda + H_0$  (see equation 2.29) suggests that  $H$  and  $P$  are proportional to energy and momentum operators and that the real constant,  $\lambda$ , is proportional to the mass. To produce Hermitian operators, which the Hilbert space formalism requires for observables, the constant of proportionality must be a pure imaginary number for  $H$ ,  $P$  and  $J$  which we shall write as  $i\hbar$ , where  $\hbar$  is Planck's constant. Equation 2.28 which defines  $\lambda$  has an accompanying factor of  $i$ , The mass is therefore  $\hbar\lambda$ . Similar relationships apply in the relativistic case.

Introducing a position operator,  $Q$ , (which must be Hermitian because position is an observable quantity) and applying a space translation, generated by  $P$  gives the relation:

$$[P_i, Q_j] = \delta_{ij} \hbar \quad (2.31)$$

or:

$$[P_x, Q_x] = \hbar \quad (2.32)$$

in the one dimensional case (we shall work in one dimension in this section because further dimensions add nothing to the argument). The fact that the commutator of  $P_x$  and  $Q_x$  is not zero is characteristic of, and symbolic of, quantum mechanics - it does not contain  $\hbar$ ; there is no arbitrary constant, the reason being that  $P$  is

the generator of space translations, which we only know to be *proportional to* the momentum. Making implicit use of the position operator to define a coordinate representation gives a specific form for  $P$ :

$$P = \nabla = \frac{\partial}{\partial x} \quad (2.33)$$

so that 2.31 reads:

$$[\nabla_x, Q_x] = 1, \quad (2.34)$$

a result which is a simple mathematical relationship well known to apply to Fourier transformations.

The failure of Planck's constant to appear is because the operator  $P$  is only *proportional to* the momentum. Relying on the existence of a position operator, allows a velocity operator,  $V$ , to be defined by:

$$\frac{d}{dt} \langle Q \rangle = \langle V \rangle \quad (2.35)$$

In terms of  $V$  we have:

$$P = i\lambda V \quad (2.36)$$

$$H = i\lambda V \cdot V + E_0 \quad (2.37)$$

$$J = iQ \times \lambda V \quad (2.38)$$

Clearly  $\lambda$ ,  $iP$ ,  $iH$  and  $iJ$  are Hermitian operators proportional to mass, momentum, energy and angular momentum, respectively. It is customary to denote the constant of proportionality by  $1/\hbar$  so that the mass,  $M$ , is  $\hbar\lambda$  and the equations become:

$$-i\hbar P \equiv \tilde{P} = \hbar\lambda V = MV \quad (2.39)$$

$$-i\hbar H \equiv \tilde{H} = \frac{MV^2}{2} + \tilde{E}_0 \quad (2.40)$$

$$-i\hbar J \equiv \tilde{J} = MQ \times V, \quad (2.41)$$

together with the commutation relation, equation 2.31:

$$-i\hbar[P_x, Q_x] = [\tilde{P}_x, Q_x] = i\hbar, \quad (2.42)$$

where we have introduced  $\tilde{P}_x = -i\hbar P_x$  as the familiar Hermitian operator for the momentum observable. Although  $\hbar$  is introduced as an unknown constant of proportionality the most significant fact is that it is *non-zero*. Letting the constant be zero gives a trivial system (the energy, momentum, angular momentum and velocity are all zero) and the commutation relation becomes the trivial identity  $0 = 0$ .

### The value of Planck's Constant

What is the value of  $\hbar$ ? This is not the simple question that it appears. The operator,  $P = \nabla$  which is the generator of space translations does not have the dimensions of momentum but of  $L^{-1}$ ; similarly  $H$ , the generator of time translations, has the dimensions of  $T^{-1}$  rather than of energy. Multiplying by  $\hbar$  with units  $ML^2T^{-1}$  gives  $P$  the units of momentum,  $H$  the units of energy,  $J$  the units of angular momentum and the constant  $\lambda$  the units of mass. The value of  $\hbar$ , unlike the value of a mathematical constant depends upon the units used. Now the conventional units for M, T, L are defined in terms of the kilogram mass of platinum-iridium alloy in Paris, the frequency of one line in the spectrum of the cesium-133 atom and the speed of light (to relate time and distance units). The value of Planck's constant in SI units is  $1.054573 \times 10^{-34}$  Joule-seconds which really means that it is:

$$8.7195 \times 10^{-9} c^2 \text{kgcycles} \quad (2.43)$$

in terms of the speed of light, the reference kilogram and cycles of the line in the spectrum of the cesium-133 atom.

Can any fundamental theory be expected to predict the value of the constants in terms of the amount of platinum and iridium atoms that were chosen as the standard for the kilogram? Clearly not. Indeed a different choice of units would give  $\hbar$  a different numerical value. A system of units could be devised that had  $\hbar = 1$ ; for many purposes such a system is both popular and convenient. The speed of light ( $c = 3.000 \times 10^8 \text{m/s}$ , dimensions  $LT^{-1}$ ) and the Gravitational constant ( $G = 6.673 \times 10^{-11} \text{m}^3/\text{kg s}$ , dimensions  $M^{-1}L^3T^{-2}$ ) can be combined with  $\hbar$  to



form a natural set of units in which all three constants have a magnitude of 1. In these units we have:

$$1\text{meter} = 6.19 \times 10^{34} \sqrt{(\hbar G/c^3)} \quad (2.44)$$

$$1\text{second} = 1.85 \times 10^{43} \sqrt{(\hbar G/c^5)} \quad (2.45)$$

$$1\text{kg} = 4.59 \times 10^4 \sqrt{(\hbar c/G)} \quad (2.46)$$

No scientific theory can explain why the meter, second and kilogram should have been chosen as units of measurement. What we can say is that the equations have a non-zero constant in them which can always be set equal to 1 by a choice of units. The units of measurement can even be defined so that the equations of physics take their simplest form, without explicit arbitrary constants  $\hbar$ ,  $c$  or  $G$ .

## 2.6 Poincaré Transformations

The Poincaré transformations are the same as Galilean ones except that a Lorentz boost,  $K_i$ , replaces the Galilean boost. The Lorentz boost is characterised by:

$$x \rightarrow x' = \frac{x}{\sqrt{1 - v^2/c^2}} + \frac{vt}{\sqrt{1 - v^2/c^2}} \quad (2.47)$$

$$t \rightarrow t' = \frac{t}{\sqrt{1 - v^2/c^2}} + \frac{xv/c^2}{\sqrt{1 - v^2/c^2}} \quad (2.48)$$

An important distinction between the Galilean and the Poincaré cases is that in the former any reference frame (any vector) can be transformed into any other by a suitable combination of symmetry operations. This means that a position and velocity is equivalent to any other, and, in particular, can be transformed to one that is stationary at the origin. By contrast, timelike, null and spacelike vectors cannot be transformed into each other by a Poincaré transformation.

### 2.6.1 Commutation Relations

The commutation relations of the Poincaré group are:

$$[P_i, P_j] = 0 \quad (2.49)$$



$$[P_i, H] = 0 \quad (2.50)$$

$$[J_i, P_j] = -\varepsilon_{ijk} P_k \quad (2.51)$$

$$[K_i, P_j] = -\delta_{ij} H/c^2 \quad (2.52)$$

$$[J_i, H] = 0 \quad (2.53)$$

$$[J_i, J_j] = -\varepsilon_{ijk} J_k \quad (2.54)$$

$$[J_i, K_j] = -\varepsilon_{ijk} K_k \quad (2.55)$$

$$[K_i, H] = -P_i \quad (2.56)$$

$$[K_i, K_j] = \varepsilon_{ijk} J_k/c^2 \quad (2.57)$$

which differ from the Galilean relations for  $[K_i, P_j]$  and  $[K_i, K_j]$  in particular.

### 2.6.2 Commutation Relations for Ray Representations

These are identical to the relations in equations 2.49 to 2.57. All the multiples of the identity can either be transformed away or be forced to be equal to zero. This is clearly true for all the commutators of  $J, H$ , and  $P$  from the same arguments as for the Galilean case, leaving just the commutators with  $K$  to check using the identity:

$$[[A, B], C] = [[C, B], A] + [[A, C], B] \quad (2.58)$$

1. For  $[K_i, H]$  we use:

$$[[J_i, K_j], H] = [[H, K_j], J_i] + [[J_i, H], K_j] \quad (2.59)$$

$$i\varepsilon_{ijk}[K_k, H] = i[P_j, J_i] + 0 \quad (2.60)$$

$$= i\varepsilon_{jik} P_k \quad (2.61)$$

$$\text{giving: } [K_k, H] = -P_k \quad (2.62)$$

2. For  $[K_k, P_i]$  we use:

$$[[J_i, K_j], P_i] = [[P_i, K_j], J_i] + [[J_i, P_i], K_j] \quad (2.63)$$

$$i\varepsilon_{ijk}[K_k, P_i] = \delta_{ij}[P_j, J_i] + 0 \quad (2.64)$$

$$= [P_i, J_i] \quad (2.65)$$

$$\text{giving: } [K_k, P_i] = 0 \quad (2.66)$$

As with the Galilean case, there is a simple relation between  $H$  and  $P$  which can be derived directly from the commutators of Poincaré transformations:

$$H^2 = c^2 P^2 + \mu I, \quad (2.67)$$

where  $\mu I$  is a multiple of the identity (or a function of operators of an unrelated internal symmetry group). Equation 2.67 can be confirmed by showing that  $H^2 - c^2 P^2$  commutes with all the generators of Poincaré transformations.

Unlike the Galilean case, there is no arbitrary parameter  $\lambda$  arising from the ray representations. The constant  $\mu$  in 2.67 is equivalent to the undetermined  $H_0$  of the Galilean case - it is this (rather than  $\lambda$ ) which is related to the mass in the Poincaré case. Consequently, equation 2.67 (unlike 2.29) is consistent with a real vector space.

As in the Galilean case, equation 2.67 can be combined with the time derivative of the equation of motion (equation 2.7) to give:

$$\frac{\partial^2}{\partial t^2} \Psi = (c^2 P^2 + \mu) \Psi \quad (2.68)$$

This is the Klein-Gordon equation provided  $\sqrt{\mu} = m_0 c / \hbar$ .

The special, and distinct case  $\mu = 0$  gives:

$$\frac{\partial^2}{\partial t^2} \Psi = c^2 P^2 \Psi \quad (2.69)$$

## 2.7 Real-Valued Quantum Mechanics

For the Galilean case the whole construction of quantum mechanics fails on a real vector space with no internal degrees of freedom. The constant  $\lambda$  in equation 2.28 is forced to be zero and the operators  $P$ ,  $H$ ,  $J$  become trivial.

Equation 2.67 suggests that the operator representing the energy be proportional to  $H$  and similarly that the operator representing momentum is proportional to  $P$ . However  $H$  and  $P$  are anti-Hermitian operators, while observables must be represented by Hermitian operators. For a complex vector space  $iH$  is a scalar multiple of  $H$  which is Hermitian and could therefore be an energy operator. For a real vector space no such multiple can be constructed. The closest useful operator in the real case is  $H^2$  which has negative eigenvalues and could correspond to minus the square of the energy. All of which strongly suggests that a complex vector space be used.

Therefore the motivation for using complex vector spaces<sup>2</sup> to represent states is the same as for the introduction of complex numbers into mathematics - we want a solution to  $H^2 = -1$ .

## 2.8 Propositional Analysis

This section gives a brief introduction to Jauch's propositional analysis[24] which formally describes the relationships and structure of the yes/no questions. The subject is covered in depth in the book by Belltrametti and Cassinelli [7] (and in a simpler way by Wantanabe [38]) who relates it to quantum mechanics as formulated on a complex Hilbert space. Propositional analysis is an abstract way of analysing a physical system which can be applied to both classical and quantum system; its power lies in the simple (even fundamental) way that quantum systems are distinguished from classical ones. Proposition systems are related to quantum mechanics as group theory is to symmetry operations.

**Definition 2.1 (A Proposition)** *A proposition is a 'yes-no experiment'.*

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<sup>2</sup>The operator  $J$ , defined by equation 2.3 can serve the same purpose as  $i$  provided the vector space has an extra degree of freedom.  $J$  is anti-Hermitian and the product of two anti-Hermitian operators is Hermitian. The other property of  $J$  that is required is that it commutes with any other operator - like a scalar does.



They can be applied to any physical system. We will be interested in applying them to statements about elementary particles and to the results of a state preparation. Any experiment which gives a range of numerical values can be converted to set of propositions. For a discrete set of measurement results  $x_i$  of a quantity  $X$ , *the result of an  $X$  measurement is  $x$*  is a proposition. For a continuous variable the same proposition has a mathematical meaning but a more practical proposition would be *the result of an  $X$  measurement is  $\leq x$* . Just as propositions can be made from measurement results, the set of all possible propositions about an  $X$  measurement can similarly be used to define  $X$  as a real valued function.

The relation between propositions and experimental measurements is not 1-1. Obviously a single experiment may allow more than one proposition to be evaluated. Conversely, two apparently different experimental arrangements may give equivalent propositions.

**Definition 2.2 (Equivalence of Propositions)** *Two propositions  $a, b$  are equivalent if  $a$  and  $b$  give the same probabilities of each result (yes or no) for every possible state of the system.*

For example, an  $x$ -position measurement and an  $x$ -momentum measurement are physically distinct and mutually exclusive but the propositions *the result of an  $x$ -position measurement is a real number* and *the result of an  $x$ -momentum measurement is a real number* are both the same trivial proposition  $I$  (which always gives yes). Another example would be two very different experimental arrangements for measuring momentum - very different signals could arise but the information gained would be the same, and the sets of propositions they generate would be indistinguishable.

Clearly there is not a one-one correspondence between experimental arrangements and propositions. The propositions can be considered as the equivalence classes of yes-no experiments with the equivalence relation being *indistinguishable by any state preparation*.

**Definition 2.3 (The Trivial Propositions: I and 0)** *We denote by  $I$  the trivial proposition which evaluates to yes for all the states in the system under consideration. We denote by  $0$  the trivial proposition which evaluates to no for all the states in the system under consideration.*

We will now consider some properties that the propositions may have, some are true for a wide range of useful proposition systems, others are mathematically useful, and some have a clear physical significance. Of particular importance is the property of distributivity which distinguishes classical and quantum systems and the weaker orthomodular property which remains valid for quantum mechanics.

**Definition 2.4 (Partial ordering  $\leq$ )**  *$a \leq b$  means that  $b$  is true whenever  $a$  is true. A proposition system is called a partial ordering if  $\leq$  is reflexive, antisymmetric and transitive. Full ordering would require that  $\forall a, b$  either  $a \leq b$  or  $b \leq a$ . We do not require full ordering.*

**Definition 2.5 (Poset)** *A Poset is a partially ordered set.*

We will use  $\mathcal{L}$  to denote a poset of propositions and  $\leq$  to denote the partial ordering relation. It is clear that for a single measurement giving results that are real numbers, propositions correspond to subsets of  $\mathbb{R}$  (*The result of an  $X$  measurement  $\in Y \subset \mathbb{R}$* ). The partial ordering is simply  $\subseteq$  applied to these subsets of the real numbers.

**Definition 2.6 (meet  $\wedge$  and join  $\vee$ )**

$$c = a \wedge b \text{ if } d \leq a \text{ and } d \leq b \Rightarrow d \leq c$$

$$c = a \vee b \text{ if } d \leq a \text{ or } d \leq b \Rightarrow c \leq d$$

Simply interpreted, they are the closest propositions that correspond to statements *that  $a$  and  $b$  are both true*, and *that either  $a$  or  $b$  is true*, respectively. Neither the

meet nor the join of two elements of a Poset need exist, but when they do this is the definition. For propositions related to measurements of a single variable the meet and join do always exist and are equated with the  $\cap$  and  $\cup$  acting on the subsets of the set of all possible measurement results. The interpretation of meet and join of propositions relating to measurements which are not only different, but also incompatible, is not trivial.

**Theorem 2.7** *The following are equivalent statements:*

1.  $a \leq b$
2.  $a \wedge b = a$
3.  $a \vee b = b$

**Proof.** The proof follows directly from the definitions of meet and join ■

**Corollary 2.8**  $a \wedge I = a$  and  $a \vee 0 = a$

**Definition 2.9 (complementation  $^\perp$ )** *Complementation is a mapping from  $\mathcal{L}$  to  $\mathcal{L}$  which satisfies:*

1.  $(a^\perp)^\perp = a$
2.  $a \leq b \Rightarrow b^\perp \leq a^\perp$
3.  $a \vee a^\perp = I$  and  $a \wedge a^\perp = 0$

Clearly  $I = 0^\perp$  and  $0 = I^\perp$ .

When a Poset has a complementation it is called an orthocomplemented poset and De Morgan's Laws are valid:

$$(a \wedge b)^\perp = a^\perp \vee b^\perp \quad (2.70)$$

$$(a \vee b)^\perp = a^\perp \wedge b^\perp \quad (2.71)$$



For measurements of a single variable, complementation is equivalent to the standard set-theoretic complement on the set of measurable results: if the proposition,  $a$ , corresponds to an  $X$  measurement which takes values in  $A \subset \mathbb{R}$ , then  $a^\perp$  is a proposition corresponding to values in  $\mathbb{R} \setminus A$ .

With complementation comes a definition of orthogonality:

**Definition 2.10 (Orthogonality  $\perp$ )**  $a$  is orthogonal to  $b$ ,  $a \perp b$ , if  $a \leq b^\perp$

For measurements of a single variable orthogonality corresponds to disjoint subsets of the set of measurement results.

Further properties can be defined by setting requirements for when the meet and join exist. Remember that although meet and join were defined for a poset there was no requirement that they existed. By not existing we mean not defined; to say that  $a \wedge b = 0$  or  $a \vee b = I$  means that they do exist.

## Mathematical Idealisations

There is a discrepancy between common practice in mathematical physics and physical necessity. Mathematically it is convenient to use the real numbers to represent the values of experiments. Classically, position, velocity, mass, time *etc.* are all represented as elements of  $\mathbb{R}$  including irrational numbers - this allowed the free use of those analytic results which required completeness. Physically there is no need to use irrational numbers - indeed it is not even possible to give a physical (rather than mathematical) meaning to them.

If we disallow irrational numbers from the propositions then there is a problem dealing with the meet or join of an infinite number of propositions; consider, for example,  $\cup_n [1, a_n + 1/n!]$  with  $a_0 = 1$  which would mathematically be represented by the set  $[1, e]$  with irrational number endpoints.

Even the use of exact real numbers is of questionable physical validity since parameters cannot practically be measured with absolute precision.



We will follow the normal practice in physics and represent measurements with a continuous range by the real numbers.

**Definition 2.11 (Lattice)** *A lattice is a Poset in which the meet and join always exists.*

For a single parameter there is no problem - the meet and join correspond to set-theoretic intersection and union of the sets of possible values.

For a continuous variable the lattice property requires that propositions exist corresponding to single numbers (rather than intervals) so that there must exist propositions such as *The result of an  $X$  measurement is exactly 2*, or even ... *The result of an  $X$  measurement is exactly  $\pi$* . We will accept these as valid propositions, although it may be interesting to try to avoid their use (Birkhoff and Von Neumann reference [8] have attempted this approach). Operators corresponding to these atoms would need to be Dirac delta functions which are not elements of a Hilbert space, a larger structure - a rigged Hilbert space is required to accommodate them [5, page 18].

**Definition 2.12 (Distributive Triplet)** *Three propositions,  $(a,b,c)$ , form a distributive triplet when:*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad (2.72)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad (2.73)$$

*and similarly for any permutation of  $(a,b,c)$ .*

**Definition 2.13 (Distributive Lattice)** *If all triplets are distributive then it is a distributive lattice.*

Since we are considering a lattice, the meet and join are always defined and both sides of 2.73 are always well-defined. For single measurements it was noted that meet and join always exist and correspond to the union and intersection of subsets of  $\mathbb{R}$ , it follows from set theory that they form a distributive lattice.

**Definition 2.14 (Disjoin Union  $+$ ,  $-$ )** For orthogonal propositions,  $a \perp b$ , we define special cases of meet and join:

$$1. a + b \equiv a \vee b$$

$$2. a - b \equiv a \wedge b^\perp$$

Only a weaker form of the lattice property is required for  $+$  and  $-$  to be defined;  $a \vee b$  only needs to exist for orthogonal pairs  $a$  and  $b$ , rather than for any two propositions.

**Definition 2.15 (Orthomodular)** A poset or lattice is orthomodular if:

$$a \leq b \Rightarrow b = a + (b - a) \quad (2.74)$$

$$= a \vee (b \wedge a^\perp) \quad (2.75)$$

**Theorem 2.16** A distributive lattice is orthomodular.

**Proof.** For  $a \leq b$ , using Theorem 2.7 we have:

$$a \vee (b \wedge a^\perp) = (a \vee b) \wedge (a \vee a^\perp) \quad (2.76)$$

$$= (a \vee b) \wedge I \quad (2.77)$$

$$= (a \vee b) \quad (2.78)$$

$$= b \quad (2.79)$$

■

**Definition 2.17 (Modular)** A poset or lattice is modular if:  $a \leq b \Rightarrow (a, b, c)$  is a distributive triplet for all  $c$

Therefore modularity is a weaker property than distributivity but is stronger than Orthomodularity:

$$\text{Orthomodular} \Rightarrow \text{Modular} \Rightarrow \text{Distributive} \quad (2.80)$$

Distributive lattices model classical systems while modular (but non-distributive) lattices model all discrete quantum logic. It can be shown that quantum systems with a continuous unbounded spectrum are modelled by an orthomodular system.

**Definition 2.18 (Atomic)** *An element of a lattice,  $p$ , is an atom if:*

$$0 \leq a \leq p \Rightarrow a = 0 \text{ or } a = p \quad (2.81)$$

Essentially, the ‘atoms’ correspond to the smallest possible subsets of the set of all measurement results. For a discrete set these are clearly the subsets with one element. For a continuous spectrum the atoms correspond to one point *eg.* the result of an X-measurement is exactly 2. As discussed above, only an ideal experiment can give meaning to such a result. But this idealisation is common to all mathematical representations of physics.

**Definition 2.19 (Covering Property)**  *$a$  covers  $b$  if  $a > b$ , and  $a \geq c \geq b$  implies either  $c = a$  or  $c = b$ . A lattice has the covering property if the join of any element,  $a$ , with an atom,  $t$ , not contained in  $a$  covers  $a$ .*

## Chapter 3

# Stochastic Processes and Quantum Mechanics

In this chapter quantum mechanics is described as a measurement-dependent stochastic process. It is the measurement dependence which gives rise to the non-classical behaviour. Simple models are described which can reproduce the statistics of quantum mechanics. Representing states by a vector, independent of the measurements, is shown (by heuristic arguments) to give rise to the usual complex Hilbert space representation of quantum mechanics.

This chapter complements the preceding formal theory of non-distributive lattices. Rigorous arguments show that some lattices can be represented by projections of a Hilbert space [7]. By contrast, the arguments in this chapter are not rigorous, but do give a clear physical picture which shows how a complex Hilbert space arises when one tries to model a measurement-dependent stochastic process.

The work in this chapter is original. It helps us to formulate the gravitational theory of quantum mechanics which follows. It reduces the complexities of quantum mechanics to a simple physical statement that *the stochastic process depends upon the measurement apparatus* (a contextual hidden-variable theory). This chapter, therefore, provides the physical picture to guide the work, whilst the preceding chap-



ter defines a mathematical route which requires the construction of non-distributive propositions.

### Stochastic processes

There is nothing strange about a theory that gives only probabilities. Any classical theory where the initial conditions cannot be completely defined for any reason will result in a range of possible outcomes (probabilities may be predictable but actual outcomes are not) which are determined by the initial distribution. Although initial conditions may not be known, the initial probability distribution may be known

Early attempts were made to explain quantum mechanics as a classical theory where some parameters were unknown. A particle was considered to be a classical object with a position, momentum, spin *etc.* which were all well-defined, but unknown in practice, due to our inability to measure them all accurately and simultaneously. Such theories are called non-contextual hidden-variable theories because the predictions depend upon variables which are not known, but which do not depend upon the measurement that is being made. Belinfante[6] has analysed and categorised such theories, and concludes that all the original interpretations of quantum mechanics where the hidden variables were associated with the particle and were independent of any measurement are inconsistent with the predictions of quantum theory and with experimental results. This does not apply to contextual theories where the hidden variables depend upon the measurement apparatus, nor to the Pilot wave theory of Bohm where a measurement-dependent wave guides a particle.

Experimental results confirm the standard quantum theory. There are no confirmed results which contradict the theory.

Two paradoxes are of particular significance. The Kochen and Specker paradox shows that the spin of a spin-1 particle cannot be assigned in a way that is independent of experiments and which still agrees with the predictions of quantum



theory. The other is the EPR experiment (and Bell's inequalities) which show that the predictions of quantum mechanics are not only at variance with any classical result but actually exhibit non-local effects. Measurements made at one location can affect the results of experiments that are spacelike separated (a signal would have to travel faster than light to communicate from one branch of the experiment to the other).

Some contrived theories have been constructed to allow particles to have well defined classical properties, but to still conform with quantum theory and experiment. These theories all have a non-local, measurement-dependent factor which influences the particles' properties. Bohm's theory is the best developed of these theories, and it does have some appealing characteristics.

The fundamental character of quantum mechanics is *not* that it only gives probabilities, nor that there is a limit to the accuracy of some experiments (such as simultaneous position and momentum measurement). It is rather that the *properties* of a particle depend upon the measurements that will be made - this dependence has a non-local character [6].

The following sections take some of the mystique out of quantum mechanics by showing that features characteristic of quantum mechanics can be reproduced if a classical measurement-dependent process is constructed. The main purpose of these contrived examples is to show how such processes can be modelled with elements of a vector space in a natural way.

### 3.1 A Measurement-dependent Stochastic Process

A classical model *can* exhibit the properties of quantum mechanics (including complementary variables) if the object being studied is the subject of a stochastic process which depends in some way upon the measurement itself. The following model is not in any way a model of quantum mechanics, but it does serve to demonstrate how complementary variables arise. Measurement conditions need to be both in-

compatible and also to affect the stochastic process in order to give the desired results.

For simplicity we will concentrate on the components of spin of a spin-half system. The objective is to define classical parameters which could correspond to the  $x$ ,  $y$  and  $z$  components of spin for a spin-half particle. The analogy includes:

1. Incompatibility of simultaneous measurement.
2. Probabilities of each measurable result correspond to those of quantum mechanics.
3. The measurement process also acts as a state preparation .

An original model is described below which displays all these features. It is interesting, not just for its simplicity, but because it shows how and why a Hilbert space can be used to model the behaviour when a classical model cannot.

Consider a die with pairs of opposite faces labelled  $x$ ,  $y$ ,  $z$  respectively; we will call this an xyz-die. For each pair, one face is red and the other blue; the exact pattern is not important. We define three measurement processes:

**Definition 3.1 (X-measurement)** *Until the top face shows an “ $x$ ”, shake the die. Then record the colour of the top face (red or blue).*

**Definition 3.2 (Y-measurement)** *Until the top face shows a “ $y$ ”, shake the die. Then record the colour of the top face (red or blue).*

**Definition 3.3 (Z-measurement)** *Until the top face shows a “ $z$ ”, shake the die. Then record the colour of the top face (red or blue).*

The wording is chosen to emphasise that if an  $x$  (or  $y$  or  $z$ ) already shows when an X (or Y or Z) measurement is required then the xyz-die is *not* shaken. How this would be achieved in practice is immaterial.

Assuming that the die is not loaded, then one measurement, followed by one of a different variable, will have a probability  $\frac{1}{2}$  of being red and  $\frac{1}{2}$  of being blue (using a loaded die we can work with any probabilities that add up to one). By contrast, a measurement that is repeated immediately will have a probability 1 of giving the same result and 0 of the alternative colour.

These probabilities are in full agreement with quantum mechanical measurements of  $x$ ,  $y$  and  $z$  components of spin for a spin-half particle. The analogy does not extend to measurements at other angles, and so does not possess the  $O(3)$  rotational symmetry that we see in the real world.

This is a hidden variable model where the exact mechanism of the shaking process contains the hidden variables; with knowledge of these variables we could predict exactly which way up the die would land. The reason that probabilities symptomatic of quantum mechanics occur is because these hidden variables depend upon both the initial state *and the measurement being taken*. The shaking process continues for a variable time,  $t$ , which depends upon the measurement being made and the initial state - “*until a face with the required label shows up*” - the time being zero if the required face is already showing. This simple model, is therefore, a hidden variable theory of the first kind (contextual hidden-variable theory), at least as far as the way in which the variables must be assigned.

Clearly evident in the model is that the  $y$ -colour has no well-defined value when an  $x$  is showing. It is not the case that the  $y$ -colour is unknown, but like in quantum mechanics, the state that shows an  $x$  has no well-defined  $y$ -colour. The  $x$ -colour and the  $y$ -colour are therefore complementary.

The die model is clearly not a model of an elementary particle; it is artificial - indeed contrived, but it does show how even a simple classical model can be devised to exhibit the probabilities associated with complementary variables.



## A Continuous-Variable Model, Dependent upon Boundary Conditions

In the xyz-die model, the results of a measurement were discrete (either red or blue) and the hidden variable, the shaking process depended upon both the measurement and the initial state. Another interesting example is of a stochastic process where the results form a continuum, and where the probability depends upon both a hidden variable *and an extra boundary condition*. In principle, the boundary condition could be linked to the measurement process.

Consider the transverse displacement,  $y$ , of a string lying along the  $x$ -axis, one end of which is shaken according to  $y(0,t) = A \sin(\omega t)$ . We can measure the  $y$  position at some point  $x$ . Without knowing the time, only a probability distribution for  $y(x)$  will be found; here the time is the hidden variable. Similarly the velocity  $\dot{y}(x)$  could be measured and another distribution found. Clearly both of these measurements depend upon the forced oscillation at  $x = 0$  *and also upon the boundary condition* at the other end,  $x = a$ ; for example  $y(a) = 0$  or  $dy(a)/dx = 0$ . These two possible boundary conditions at  $x = a$  are clearly incompatible with each other. This is still a normal classical problem, but some of the nature of quantum mechanics can be introduced by defining position and velocity measurements that are incompatible *ie.* the position is measured by clamping the string, while the velocity is measured by a process that requires the string to move. We now have two stochastic complimentary measurements dependent on the hidden variable  $t$  and also upon the boundary condition at  $x = a$ .

In principle, the measurements could be defined to depend upon the boundary conditions also.

## 3.2 Modelling a Stochastic Process

Jauch and Beltrametti[24, 7] justify the use of a Hilbert space to represent an orthomodular propositional calculus using mathematical theorems. The mathematical



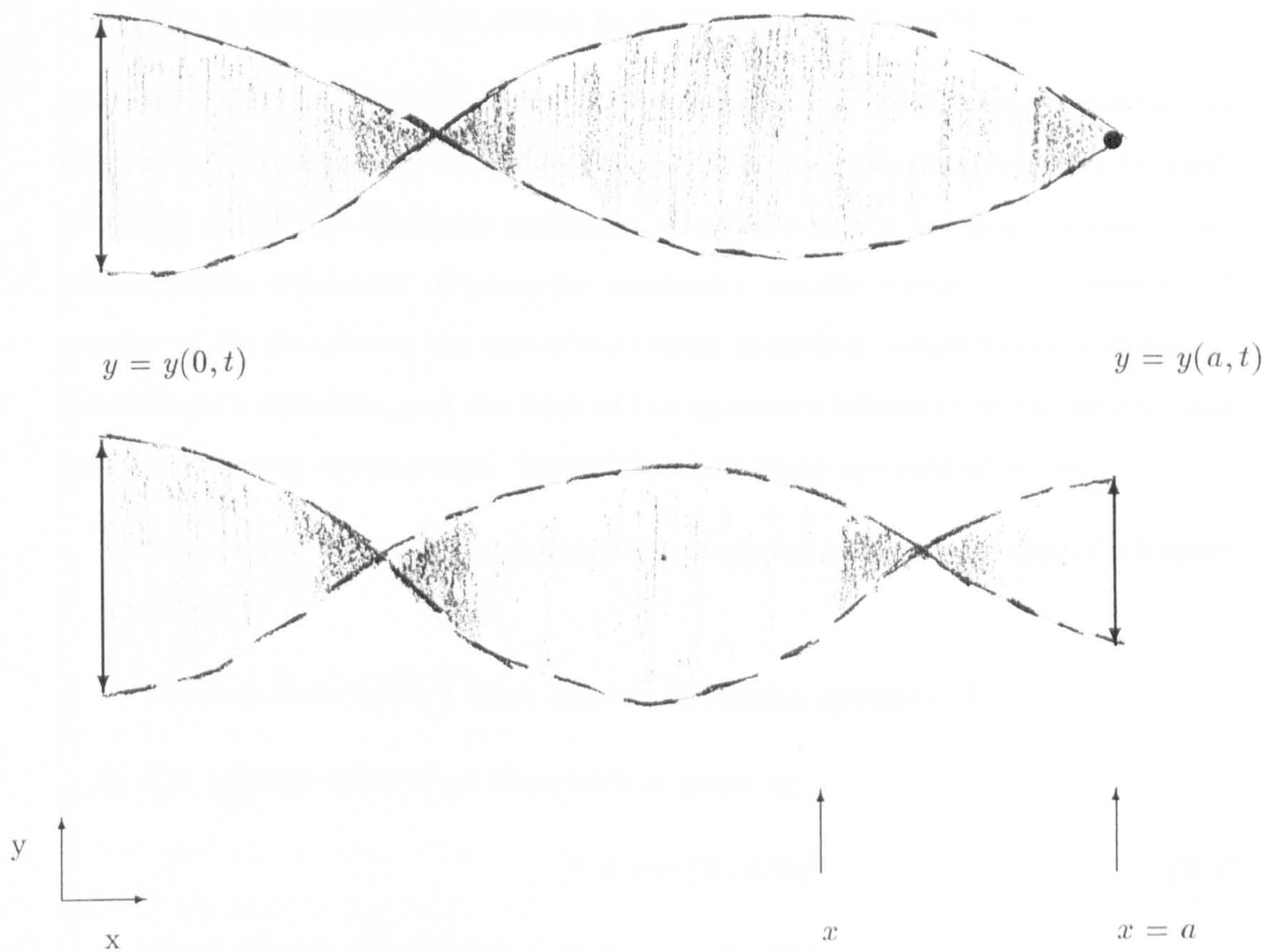


Figure 3.1: A Stochastic Process Dependent upon a Boundary Condition. Measuring the displacement of the string at a position  $x$  gives a range of results, but the range depends upon the boundary conditions at both ends.

basis has been further strengthened by more recent results[21]. This original analysis takes a more pragmatic approach. The objective is to formulate a mathematical model of a measurement-dependent stochastic process and to do so by describing a state in a measurement-independent way. There are two related reasons for doing so;

1. This is the closest to the classical situation where states are presumed to exist *regardless* of measurements that may or may not take place.
2. This is how people have chosen to describe quantum mechanics.

We will examine the consequences of representing the state of a particle by an element of a vector space. A space of the smallest possible dimension will be used. It will be found that quantum mechanics then follows as a necessary consequence. An axiomatic treatment of quantum mechanics usually starts from a very small number of simple axioms, the rest of the theory, including complementary variables, Schrödinger's equation, and the form of the operators follows from the axioms and certain symmetry assumptions. Essentially the axioms are equivalent to:

1. The state of a particle (or system) is represented by a unit element of a Hilbert space,  $\Psi$ .
2. For each observable,  $a$  there exists a Hermitian operator,  $A$ .
3. The average value of an observable is given by:

$$\langle a \rangle = (\Psi, A.\Psi) \quad (3.1)$$

where  $(\Psi_1, \Psi_2)$  is the inner product on the Hilbert space.

We will show that these axioms arise naturally as a consequence of using elements of a vector space (of the smallest possible dimension) to describe a stochastic process which is dependent upon both the measurement and the initial conditions. A very simple linear model using vectors to represent the states of a particle is seen

to be sufficient for a full description. The xyz-die example can be used to illustrate the process.

### A Simple Stochastic Process

We describe the state of an ordinary die by a mathematical object (call it  $\Psi$ ) that can be used to predict the probabilities of any measurement result. Therefore  $\Psi$  is a function of the current state and independent of any subsequent measurement. We are resigned to predicting only probabilities not certain results. We look for  $\Psi$  to be a complete description in that it can predict anything that can possibly be predicted - this too is what we would expect to be able to do from classical mechanics.

Let us list the probabilities of possible outcomes for a traditional 6 sided die shaken and rolled in the usual way:

$$\Psi = \begin{pmatrix} 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \\ 1/6 \end{pmatrix} \quad (3.2)$$

Each row corresponds to a different outcome (1,2,3,4,5,6) and the elements give the probability of that outcome. All positions of the die are equivalent as far as predicting possible outcomes, so there is only one  $\Psi$  to represent any possible state of the die. The format is deliberately suggestive of a vector. In the classical sense there is no more or less to be said.

### A Discrete Measurement-dependent Model

Returning to the description of the xyz-die, we note that there are six initial states (red and blue for each of x,y,z), represented by  $\mathcal{I} = \{x_r, x_b, y_r, y_b, z_r, z_b\}$ , three measurement operations, and three types of result for each, represented by {certainly



red, certainly blue, 50/50 chance of red or blue}. There is no requirement for the X,Y,Z measurements to be related in any way (the labelling of the axes is arbitrary - therefore what is true for X,Y,Z should also be true if they were relabelled Y,Z,X), nor is there a non-trivial continuity requirement, since the operations of measurement act on a discrete set of possible states. The states could each be represented by a table of the probabilities obtained for each of the three measurements:

$$x_r = \begin{pmatrix} 1 \\ 0 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \quad x_b = \begin{pmatrix} 0 \\ 1 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}, \quad y_r = \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \\ 0 \\ 0.5 \\ 0.5 \end{pmatrix}, \quad \text{etc.} \quad (3.3)$$

Thus a state is represented by a six component object with each of the six coefficients giving the probability of a different measurable result being obtained. There is no concept of a measurement that is “almost an X-measurement” and so the above description, essentially a look-up table, is adequate. Although the initial states are described independently of the measurements (they are listed), they explicitly contain information about each type of measurement that can be made even though they cannot be made simultaneously and despite the fact that after any measurement only two states are possible. It might be hoped to reduce the number of parameters in the model from six to only one or two. We will show that using one parameter is inadequate, while two parameters is not only sufficient but has the same structure as quantum mechanics.

### A one parameter model

We will try to construct a vector space to represent the states, using as few dimensions as possible, but where the coefficients can give the probability of different measurement outcomes, with a mapping given by some as yet unknown function



$f$ . Let  $\psi = \hat{x}_r$  denote a state that definitely gives an  $x$ -result of red. Our first attempt will consider  $\hat{x}_r$  as a base vector of a *one* dimensional vector space. If an  $x$ -measurement is made the result is either red (corresponding to  $\hat{x}_r$ ) or not-red. Let us therefore attempt to denote each state by some multiple of  $\hat{x}_r$ : We immediately have:

$$x_r \rightarrow \hat{x}_r = 1.\hat{x}_r \quad (3.4)$$

with a coefficient of 1 and a probability of 1, hence  $f(1) = 1$  Let us write:

$$x_b \rightarrow a\hat{x}_r \equiv \hat{x}_b \quad (3.5)$$

Now  $x_b$  has zero probability of being  $x$ -red and hence  $f(a) = 0$ . We cannot determine  $a$  yet. Continuing, we try to write the state  $y_r$  as a multiple of  $\hat{x}_r$ ;

$$y_r \rightarrow b\hat{x}_r = b/a\hat{x}_b \equiv \hat{y}_r \quad (3.6)$$

assuming  $a \neq 0$ . Now  $y_r$  has a 50% probability of being  $x$ -red and hence  $f(b) = 0.5$  and a 50% probability of being blue hence  $f(b/a) = 0.5$  also. If we require a linear model then we must allow:

$$\hat{y}_r = (1 - \lambda)b\hat{x}_r + \lambda b/a\hat{x}_b \quad (3.7)$$

for any  $\lambda$ . Note that we are assuming that the numbers  $a, b, \lambda$  are associative. This gives the absurd result that  $f(\lambda b/a) = 0.5$  for all  $\lambda$ . The special case of  $a = 0$  which implies  $f(0) = 0$  gives the contradiction:

$$\hat{x}_b = 0.\hat{x}_r = \lambda 0.\hat{x}_r = \lambda\hat{x}_b \quad (3.8)$$

which gives  $f(\lambda) = 1$  for all  $\lambda$ .

So we see that with associative numbers, and the requirement for a linear structure, it is not possible to define a function  $f$  that gives the probabilities consistently on a 1 dimensional representation.

### A two parameter model

We proceed as above, with  $\Psi = \hat{x}_r$  denoting a state that definitely gives an  $x$ -result of red. The other possible outcome of an  $x$ -measurement is blue. This time let us denote this by an independent object  $\hat{x}_b$ ; we could justify this by pointing out that for a state, before being measured “NOT  $x_r$ ” does not imply  $x_b$  since  $y_r$  is another possibility. Now we make the identifications:

$$x_r \rightarrow \hat{x}_r = 1.\hat{x}_r \quad (3.9)$$

with a coefficient of 1 and a probability of 1, hence  $f(1) = 1$  and:

$$x_b \rightarrow \hat{x}_b = 1.\hat{x}_b \quad (3.10)$$

Assuming a linear structure gives:

$$\hat{x}_r = 1.\hat{x}_r = 1.\hat{x}_r + 0.\hat{x}_b \quad (3.11)$$

hence  $f(0) = 0$ . To proceed we require that  $y_b$  *etc.* can also be expressed as linear combinations of  $\hat{x}_r$  and  $\hat{x}_b$ . We can then express any state as a two dimensional vector  $\Psi$  over the field  $F$  (which is so far unspecified). The objects  $\hat{x}_r$  and  $\hat{x}_b$ ; act as orthogonal unit vectors which represent results of ‘red’ and ‘blue’, respectively, for an  $X$ -measurement. Each component is related by the function  $f$  in some, as yet unspecified, way to the probability of that outcome:

$$\Psi = a_x \hat{x}_r + b_x \hat{x}_b \quad (3.12)$$

This composition of the vector is clearly related to one particular measurement, (the  $X$ -measurement in this case). For  $\Psi$  itself to be independent of the measurement (as we naturally require, and because we are assuming that  $y_r$  *etc.* can be written in terms of  $\hat{x}_r$ ) it must be possible to describe the very same vector in terms of  $Y$  and  $Z$  measurements *ie.*:

$$\Psi = a_x \hat{x}_r + b_x \hat{x}_b$$

$$\begin{aligned}
&= a_y \hat{y}_r + b_y \hat{y}_b \\
&= a_z \hat{z}_r + b_z \hat{z}_b
\end{aligned} \tag{3.13}$$

where the vectors in each of the pairs  $\{\hat{x}_r, \hat{x}_b\}, \{\hat{y}_r, \hat{y}_b\}, \{\hat{z}_r, \hat{z}_b\}$  are independent. The remaining ingredient is a function,  $f(a)$ , which maps the coefficients to the probabilities. It must be single-valued or it would not predict probabilities; so we require:

$$f : (a \in F) \rightarrow P(X \text{ being red}) \in [0, 1] \tag{3.14}$$

It remains to be seen if this vector space description can give the required probabilities and, if so, what implications there are. Later we will consider whether  $a_x$  can be a real or complex field; it could even be possible for  $a_x$  to be an element of the quaternion division ring. The following two theorems have already been established:

**Theorem 3.4**  $f(1) = 1$

**Proof.** Since we have defined  $\hat{x}_r$  to represent a result where an X measurement gives red, then  $\Psi = \hat{x}_r$  must correspond to having an X result of red only, *ie.* with certainty. The coefficient of  $\hat{x}_r$  is one, and the probability is one. ■

**Theorem 3.5**  $f(0) = 0$

**Proof.** Since we have defined  $\hat{x}_r$  to represent a result where an X measurement gives red, then  $\Psi = \hat{x}_r$  must correspond to not having an X result of blue, *ie.* it is impossible. The coefficient of  $\hat{x}_b$  is zero and the probability is zero. ■

We can now get further information about the function  $f$ :

**Proposition 3.6**  $f(a) = f(|a|)$

**Justification** Consider  $\Psi = a_x \hat{x}_r + b_x \hat{x}_b$  and change the basis by:

$$\hat{x}_r \rightarrow \hat{x}'_r = e^{i\phi} \hat{x}_r \tag{3.15}$$

$$\hat{x}_b \rightarrow \hat{x}'_b = \hat{x}_b \tag{3.16}$$



Now  $\Psi = a_x e^{-i\phi} \hat{x}'_r + b_x \hat{x}_b$ . If we attach any meaning to this expression it must be that  $b_x$  still gives the probability of having a blue measurement for X. Also note that  $\hat{x}'_r$  depends only on  $\hat{x}_r$ , a red outcome for an X measurement, with a coefficient of  $a_x e^{-i\phi}$ , which by our construction must represent the probability and give the same value for  $a$ .

Therefore:  $\forall \phi, f(e^{i\phi} a) = f(a)$  ■

The weakness in the justification, for the xyz-die, is that  $\Psi$  can be written as other linear combinations of unit vectors which have no physical counterpart; it could therefore be argued that  $e^{i\phi} \hat{x}_r$  also has no physical meaning. For the realistic case of a particle with spin-half, different bases correspond to measurements of the spin in other directions. They have a physical interpretation so it is reasonable to give a physical interpretation to any base vectors.

The use of a complex phase factor,  $e^{i\phi}$ , was for convenience; for a real vector space there would only be a factor  $\pm 1$ , while for a quaternion space the  $j$  and  $k$  generators would also have to be included. We will not consider the quaternion case any further, but will leave open the option of having a real vector space.

**Theorem 3.7 (quadratic probability function)**  $f(a) = |a|^2$

**Proof.** It was postulated in equation 3.13: that there were at least three different ways of expressing  $\Psi$  which were physically meaningful, corresponding to the three different and mutually exclusive ways in which the die can be measured (although mathematically there are a whole range of orthonormal transformations of bases). These three at least must satisfy the condition for the total probability being 1:

$$f(|a_i|) + f(|b_i|) = 1 \quad (3.17)$$

where  $i = x, y$  or  $z$ . In any basis the amplitude of  $\Psi$  is:

$$|a_i|^2 + |b_i|^2 = 1 = |\Psi|^2 \quad (3.18)$$



These equations together with  $f(0) = 0$  and  $f(1) = 1$  are *insufficient* to uniquely determine  $f(s)$ . It will be satisfied by any function of the form:

$$f(s) = \frac{1}{2} + f_0(s^2 - \frac{1}{2}) \quad (3.19)$$

where  $f_0$  is any odd function satisfying  $f(\frac{1}{2}) = \frac{1}{2}$ . The simplest, non trivial, choice is  $f_0(r) = r$  giving  $f(a) = |a|^2$ ; but another suitable choice would be  $f_0(r) = r^3$  giving a probability function of  $f_3(a) = 3|a|^2 - 6|a|^4 + 4|a|^6$  which satisfies all the constraints defined so far. There are two ways to get a unique function;

1. To require each component of the vector to have an invariant meaning in its own right, so that for  $a$ :

$$\Phi = a_x \hat{x}_r \quad (3.20)$$

In a new basis:

$$\Phi = \alpha \hat{y}_r + \beta \hat{y}_b \quad (3.21)$$

The conditions for any  $a_x$  are now:

$$f(|\alpha|) + f(|\beta|) = f(|a_x|) \quad (3.22)$$

$$|\alpha|^2 + |\beta|^2 = |\Phi|^2 = |a_x|^2 \quad (3.23)$$

The only continuous function  $f$  which satisfies these conditions for all values of  $a_x$  is  $f(a) = |a|^2$ .

2. To consider a measurement that can have three or more mutually exclusive outcomes; the constraints are then:

$$f(|a_i|) + f(|b_i|) + f(|c_i|) = 1 \quad (3.24)$$

$$|a_i|^2 + |b_i|^2 + |c_i|^2 = 1 = |\Psi|^2 \quad (3.25)$$

The only continuous function which satisfies these constraints is  $f(a) = |a|^2$ .

The latter argument is essential for vector representations with more than 2 dimensions, while the former is a natural requirement given that vector representations are being used. Both options are reasonable and give the same result. ■

**Theorem 3.8 (Hermitian operator)** *The measurements are represented by Hermitian operators. The possible values of a measurement are the eigenvalues and the average value is given by:  $\langle a \rangle = (\Psi, A.\Psi)$*

**Proof.** We will consider a simple example of an X-measurement of spin of a spin-half particle. Choosing a basis appropriate for the measurement  $\hat{x}_u$  (with an  $x$ -spin of  $+\frac{1}{2}$ ) and  $\hat{x}_d$  (with an  $x$ -spin of  $-\frac{1}{2}$ ). Using the result of Theorem 3.7:

$$\Psi = a_x \hat{x}_u + b_x \hat{x}_d \quad (3.26)$$

$$\langle a \rangle = \frac{1}{2}P(X \text{ up}) + (-\frac{1}{2})P(X \text{ down}) \quad (3.27)$$

$$= \frac{1}{2}|a_x|^2 + (-\frac{1}{2})|b_x|^2 \quad (3.28)$$

$$= (\Psi, A.\Psi) \quad (3.29)$$

$$\text{where: } A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad (3.30)$$

Clearly  $A$  is Hermitian and, being diagonal, the eigenvalues are seen to be the measurable results  $\pm\frac{1}{2}$ . The construction is trivial in the appropriate basis, but the eigenvalues,  $a$ , and the Hermitian property are basis independent. ■

**Theorem 3.9 (ray representation)** *This is a Ray representation.*

A ray representation is one where each physical state is represented by more than one element (of the vector space).

**Proof.** Any result can be given by:

$$\langle a \rangle = (\Psi, A.\Psi) \quad (3.31)$$

for some  $A$ . The same result is also obtained for  $e^{i\phi}\Psi$  Therefore the vectors  $e^{i\phi}\Psi$  for all real numbers  $\phi$  give identical physical results. ■

Note that this is not a change of basis - there is no change to  $A$ . It is an entire set of physically indistinguishable vectors. For a real vector space the set has

just two elements  $\pm\Psi$ , while for a complex vector space there is a continuous set parameterised by  $\phi$ .

**Theorem 3.10 (unitary symmetry)** *Physical results are unchanged by unitary transformations of the vector space:*

**Proof.** Physical results are obtained from a scalar product  $(\Psi, \Phi)$ , such as in equation 3.31. This is invariant under a unitary transformation. Furthermore, unitary transformations are the only *linear* transformations which preserve the inner product. Anti-linear, anti-unitary operators exist which also preserve the inner product.

■

Representing symmetry operations by unitary operators is, therefore, a consequence of using a vector space to model the probabilities; it is not a separate assumption. This is Wigner's theorem [41].

The framework established is equivalent to quantum mechanics. For physically realistic cases, possessing the correct symmetries for spacetime, the actual form of the operators can be derived from the symmetry operations of space and time. In fact the identification of operators requires a vector space over the field of the complex numbers, (the real numbers are not adequate 2.7).

### The probability function and scalar invariants

A probability is a real scalar. If we choose to represent a state by a vector which can be described with respect to a variety of different bases then there is a limited number of ways to construct an invariant scalar from the vector. Clearly a probability should not depend upon the basis used to represent the state. This leads naturally to the scalar product  $(\Psi, \Psi)$  or the scalar product of the vector and another vector  $(\Psi, \Phi)$ . The scalar product is, of course, a quadratic function of the coefficients.

It is interesting to note that had we chosen an operator to represent a state, then the requirement for a scalar invariant would have suggested either the trace



or the determinant as suitable constructions. However only the trace is a linear function of operators:

$$\text{tr}(A) = \text{tr}(P_x A + P_y A) = \text{tr}(P_x A) + \text{tr}(P_y A) \quad (3.32)$$

where  $P_x$  and  $P_y$  are the  $x$  and  $y$  projection operators respectively. No such decomposition is possible for the determinant. The description of states by operators therefore leads to the use of the trace function to extract probabilities - which is the density matrix formulation of quantum mechanics.

### Gleason's theorem

The quadratic probability function theorem 3.7 and Gleason's famous theorem *...Measures on the Closed Subspaces of a Hilbert Space*[16]...are in fact the same despite the fact that they are expressed in a rather different language.

Gleason assumes a Hilbert space structure and looks for *measures*,  $\mu$ , such that

If  $\{A_i\}$  is a countable collection of mutually orthogonal subspaces having closed linear span,  $B$ , then:

$$\mu(B) = \sum_i \mu(A_i) \quad (3.33)$$

It is clear that for every positive self-adjoint operator,  $T$  of the trace class-

$$\mu(A) = \text{tr}(TP_A) \quad (3.34)$$

(where  $P_A$  denotes the orthogonal projection on  $A$ )  $\mu$  defines a measure on the closed subspaces. Gleason goes on to prove that every measure can be expressed in this way for dimensions greater than 2.

The probability functions which can be broken down into the sum of constituent parts (which we require) are a subset of these measures - since a probability must be positive and less than 1 (bounded). Our condition that more than two dimensions is required is consistent with Gleason's proof.



## Conclusion

We have shown that the familiar structure of quantum mechanics is suitable for describing a measurement dependent stochastic process in a measurement independent way. Indeed, it is suggested that the familiar structure is a unique way of modelling such processes using vectors.

## Counter-examples

The preceding analysis demonstrates that quantum mechanics, as formulated on a complex Hilbert space, follows if states of a measurement-dependent stochastic process are modelled by a vector of greater than two dimensions. The proofs that construct a real, complex or quaternion Hilbert space from an orthomodular lattice make the same two assumptions that the number of possible outcomes (the dimension) is greater than two and that a vector representation is possible.

Mielnik [29] has produced some counterexamples which show that a vector representation might not be possible. The essence of his argument is that the equation:

$$P_{\psi,\phi} = |(\Psi, \Phi)|^2 \quad (3.35)$$

(which gives transition probabilities from one state to another) is independent of the mapping from propositions to subspaces of a Hilbert space. He proceeds to construct a fictional example which shows that vectors cannot be found with the required transition probabilities:

*“...Someone looked at a small spherical glass bubble; inside there was a drop of liquid. The drop occupied exactly half the bubble in the shape of a hemisphere. He was able to introduce a thin flat partition dividing the interior of the bubble into two equal volumes. He tried to do this so that the drop would become split. However, the drop exhibited a quantum behaviour: instead of dividing into two parts the drop jumped and occupied the space on only one side of the partition. He repeated the attempt*

*and obtained a similar result. He began to observe the phenomenon and discovered that each time the partition is introduced the drop chooses a certain side with a definite probability. This probability depends upon the angle between the partition and the initial surface of the drop. If the drop occupied a hemisphere  $s$  and the partition forces it to choose between  $r$  and  $r'$  the probabilities of the transition into  $r$  and  $r'$  are proportional to the volumes  $s \cap r$  and  $s \cap r'$ ..”*

He continues:

*“...He was struck by the analogy between positions of the drop and quantum states and between the partition and the macroscopic measuring apparatus. He wanted to formulate the quantum theory of this phenomenon, but he realised that he could not use Hilbert spaces: the space of states of the drop was not Hilbertian.”*

Because the transition probabilities for his example are  $1 - \theta/\pi$ , where  $\theta$  is the angle between the planes, - while a spin-half system has a transition probability of  $\cos^2(\theta/2)$ . Unfortunately his ingenious example is in only two dimensions. The lattice of propositions and the symmetry of the example are identical to those of a spin-half system. Section 3.2 showed how Mielnik's example can be represented with vectors. The same vectors are assigned as for spin-half particles (which requires a complex vector space). The extra freedom in two dimensions is used to define a new function  $f_0(|s|^2)$  from equation 3.19. Conventional quantum mechanics has  $f_0(|s|^2) = |s|^2 - 1/2$ , while Mielnik's example requires  $f_0 = 1/\pi \cos^{-1}(2|s|^2 - 1)$ , which replaces 3.35 by:

$$P_{\Psi,\Phi} = 1 - \theta/\pi \quad (3.36)$$

So we see that Mielnik's example can be represented by projections of a Hilbert space, provided that a different choice of norm is used for the transition probabilities. His example does not show that a vector representation is impossible.



The xyz-die, which conveniently gave the model as for  $x$ ,  $y$  and  $z$  components of spin for a spin-half system, can be extended to give counter-examples where vectors cannot be used to represent the states. The xyz-die required three pairs of orthogonal vectors such that the transition probability between one vector and its orthogonal partner was 0 while with any other vector it was  $\frac{1}{2}$ . The required vectors are, as for spin-half:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad (3.37)$$

It can be seen that the vectors in each pair are orthogonal  $(\Psi, \Phi) = 0$ , and that  $|(\Psi, \Phi)| = 1/\sqrt{2}$  for vectors from different pairs.

It should therefore be clear that for the xyz-die, the complex-valued vectors are both necessary and sufficient to represent the states. The simpler case with two pairs of faces (an appropriately labelled octahedron or a square cylinder without ends, perhaps called an xy-die) could be described by real vectors. By contrast, if a dodecahedral or icosahedral die was used, then six or ten pairs of vectors, respectively, would be required (the dodecahedral die would be an xyzuvw-die). Each vector would still be two dimensional because there are only two outcomes to any measurement; and each pair of vectors would still need to be orthogonal because a red could not change to a blue given the rules of shaking, but by the same rules, the probability of  $\frac{1}{2}$  must be obtained for a transition from an element of one pair to one from a different pair. This is not possible with complex vectors (or even quaternion vectors).

The dodecahedral and icosahedral die offer a counter example along the lines that Mielnik suggests. A vector representation is not possible, because the number of independent states is just too large. This leaves an open question: *Why does quantum mechanics using a complex Hilbert space describe Nature?* Perhaps it does not. It could be that the role of filters, together with the symmetries of spacetime



limit the probability spaces to those which can be described by a complex vector space. By giving an explanation for the origin of complementary observables, this work offers a way to answer this question. An alternative resolution may be that some observables cannot be represented by a complex Hilbert space. How would such phenomena manifest themselves?

## Chapter 4

# General Relativity and Quantum Mechanics

A careful consideration of Einstein's theory of relativity shows that the mathematics does not match the common conception of the theory. Small distortions of a flat Euclidean spacetime are able to explain all known experimental results. Indeed, a linear theory of gravitation is sufficient for explaining all known experimental results. Theoretical work on black holes does go further and examines highly distorted space, Yet most work is still with flat space that has been deformed. However, the mathematical formulation, which is definitive of the theory, is far richer. We will examine some of these features in the following sections. In doing so we will go far beyond the picture of spacetime which was envisaged when the theory was developed; yet all this work remains within the classical theory of general relativity as originally defined by Einstein's equations.

## 4.1 Non-linearity in General Relativity and Quantum Mechanics

Einstein's theory is non-linear. Given solutions,  $g_{\mu\nu}$ , and  $g_{\mu\nu}'$  then  $\alpha g_{\mu\nu} + \beta g_{\mu\nu}'$  will not in general be a solution. Although there are useful linear approximations to Einstein's equations they are only approximations. The gravitational field is affected by mass/energy of all forms and it also carries energy. It can therefore be considered as a source for itself (see for example [39] p165) - an intrinsically non-linear affect. Alternatively, an examination of the equations ( $R_{\mu\nu} = 0$  for the vacuum) shows that they are quadratic in the metric and its first derivatives since:

$$R_{\beta\delta} = R^{\gamma}_{\beta\gamma\delta} \quad (4.1)$$

where:

$$R_{\nu\beta\gamma\delta} = g_{\nu\alpha} \left( \partial_{\gamma} \Gamma^{\alpha}_{\beta\delta} - \partial_{\delta} \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta} \Gamma^{\mu}_{\beta\gamma} \right) \quad (4.2)$$

and:

$$\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2} g^{\kappa\lambda} (\partial_{\lambda} g_{\mu\nu} - \partial_{\nu} g_{\mu\lambda} - \partial_{\mu} g_{\nu\lambda}) \quad (4.3)$$

Quantum mechanics is also non-linear. The basic equations (Schrödinger, Klein-Gordon, Dirac) are linear in the wavefunction,  $\Psi$ . If  $\Psi$  and  $\Psi'$  are solutions to one of the wave equations then  $\alpha\Psi + \beta\Psi'$  is also a solution. The non-linearity lies in the measurement process, or to be more specific in the rule that the only possible results of a measurement are the eigenvalues of a corresponding Hermitian operator. When the eigenvalues form a discrete spectrum, then the existence of measurable results  $\lambda_1$  and  $\lambda_2$  with eigenvectors  $\Psi$  and  $\Psi'$  does not imply that the linear combination  $\alpha\lambda_1 + \beta\lambda_2$  is an eigenvalue nor that  $\alpha\Psi + \beta\Psi'$  is an eigenvector at all - despite the fact that it is a valid solution of the wave equations!



## 4.2 The Relation Between Curvature and Topology

Einstein's equations can be considered as equations for the components of the metric tensor which gives the information needed to measure intervals in any local choice of coordinate system. From the metric tensor and its derivatives the curvature can be calculated in the form of the Riemann tensor which has 0, 1, 6 and 20 independent components in 1,2,3 and 4 dimensions respectively.

Topology is a global property; manifolds with the same topology can be mapped into each other by a continuous 1-1 transformation *ie.* with no cutting, pasting or overlapping. For example the sphere  $S^2$  with a point removed (say the North pole) and the plane  $\mathbb{R}^2$  - the transformation being a stereographic projection. If the point had not been removed there would be no such transformation,  $S^2$  and  $\mathbb{R}^2$  are topologically different.

The relationship between the metric and the topology is fascinating; they constrain each other but neither is sufficient to determine the other. For example two spaces can have the same metric locally but be topologically distinct. A simple example in two dimensions is the plane  $\mathbb{R}^2$  and the cylinder  $S^1 \times \mathbb{R}$  both are locally flat as far as the metric is concerned; there are no local measurements that can be carried out to distinguish the two cases. Although our common perception of a cylinder is as a 2D surface necessarily embedded in a 3D space, there are other ways of describing a cylinder without using a third dimension; it can be regarded as a rectangle with two opposing sides identified (see figure 4.1).

In a similar way, flat tori can be defined by identifying opposite faces of a rectangle.

The earlier example of a sphere with a point removed and  $\mathbb{R}^2$  shows two spaces with the *same* topology but *different* metrics. An observer on the sphere, for example, would find that the angles of a triangle did not add up to exactly 180 degrees.

For compact manifolds in two dimensions the curvature is related to the

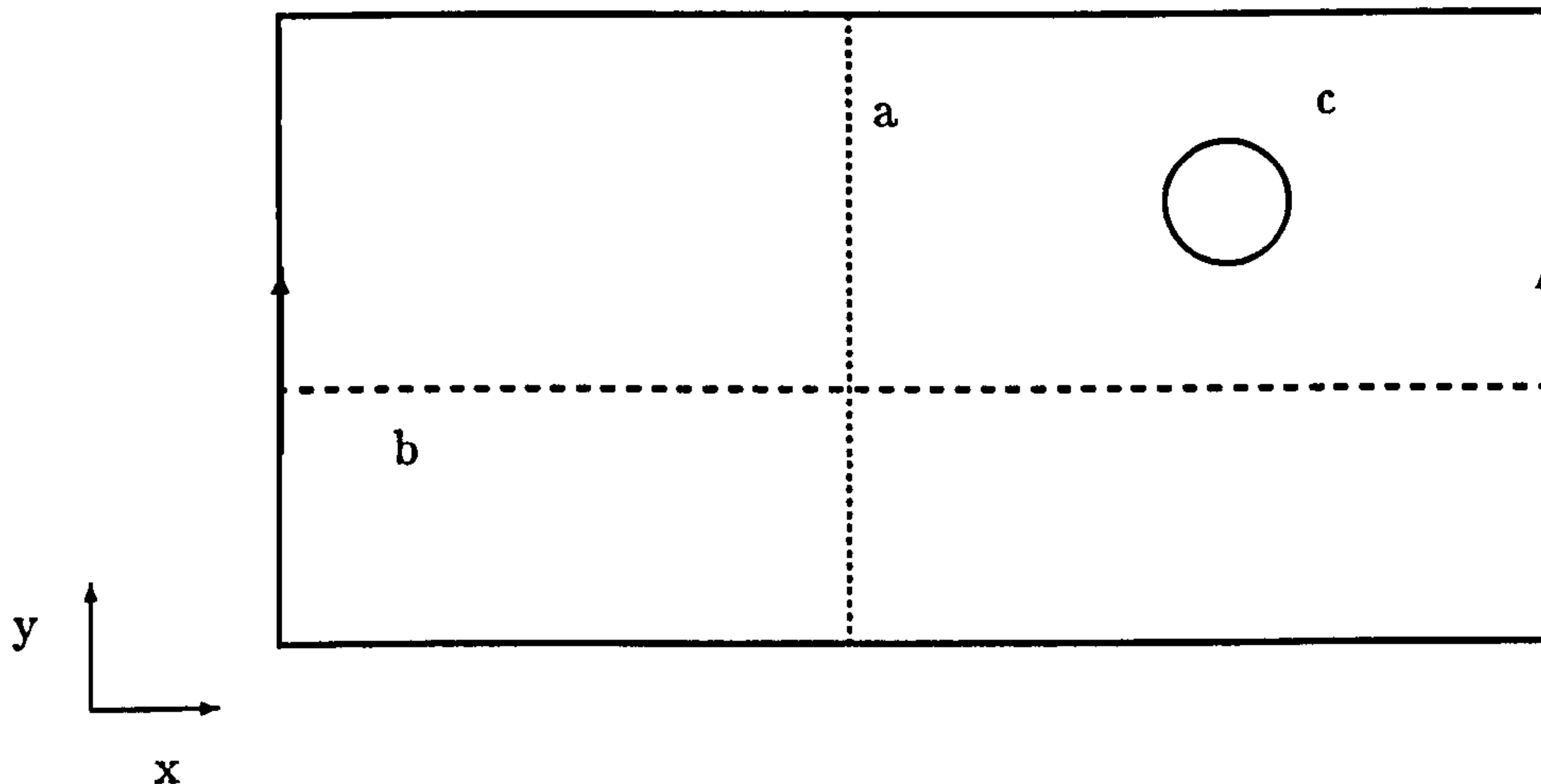


Figure 4.1: The cylinder as a rectangle with sides identified. Curve  $a$  is a straight line, curve  $b$  (which is also straight) joins itself to form a closed loop, curve  $c$  is another closed curve which, unlike  $b$ , can be contracted to a point.

Euler characteristic,  $\chi$  by:

$$\chi = \frac{1}{4\pi} \int R dV \quad (4.4)$$

The Euler characteristic is a topological measure, while the scalar curvature,  $R$ , embodies some (but not all) information about the curvature.

### 4.3 Manifolds With Closed Timelike Curves

One of the exciting features of general relativity which has enjoyed considerable debate and speculation is that it permits structures of spacetime where time itself bends around in a closed curve, *ie.* a closed timelike curve (CTC). In essence, it allows time machines. This is so astonishing and contrary to our experience that a chronology protection mechanism has been sought which would prevent such loops

occurring. To date no such mechanism has been established and we are left with the fact that the best theory we have of space and time, general relativity, admits closed timelike curves. Models with CTCs which conform to Einstein's equations (but do not model our universe) can easily be constructed in the same way that the cylinder was defined by identifying two opposing sides of a rectangle. Depicting one space and one time dimension we can construct a space with CTCs, as in figure 4.2

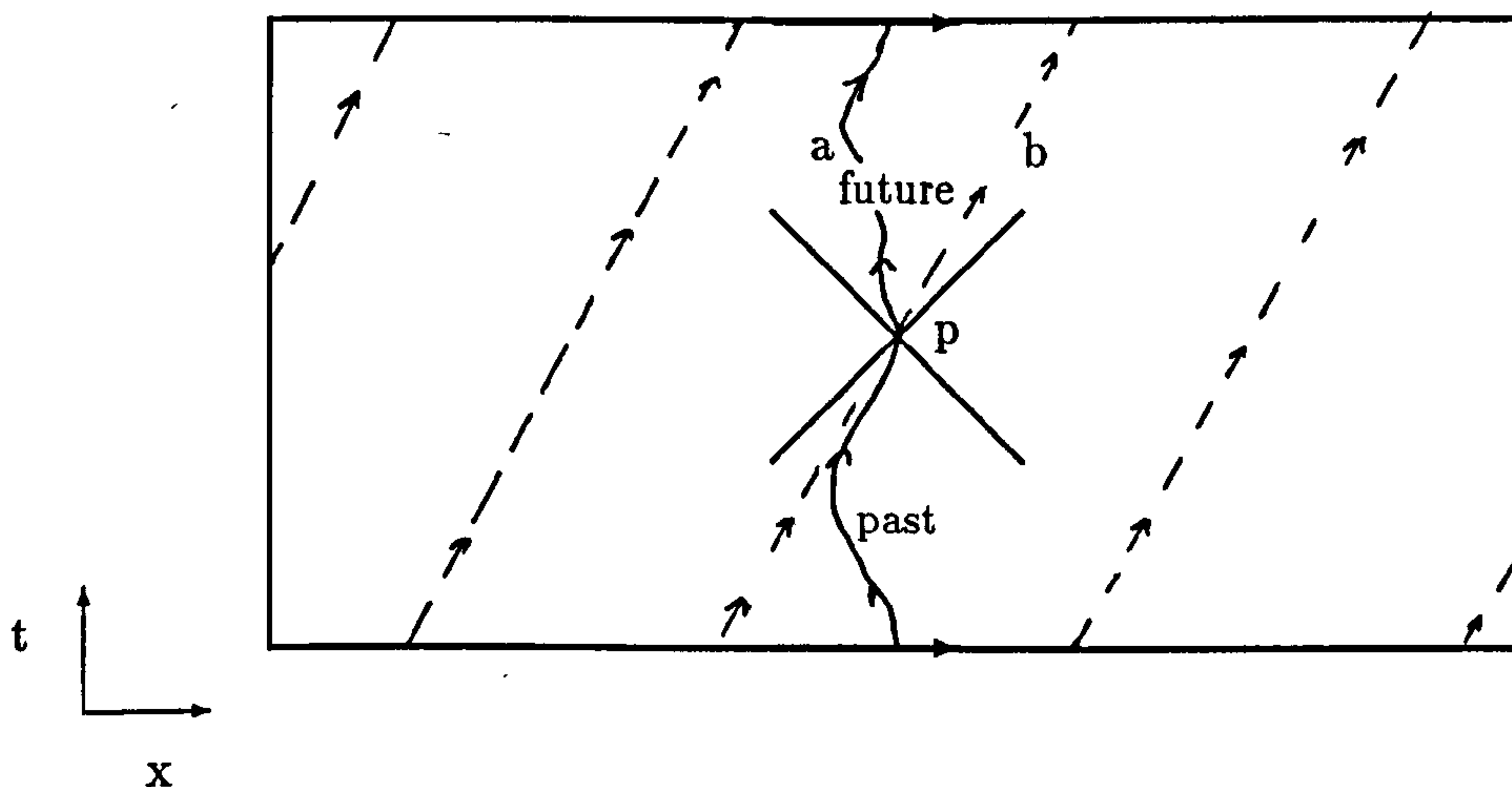


Figure 4.2: A space time with CTCs as a rectangle with sides identified. Every timelike curve through  $p$  will reappear in its own past. Trajectory  $a$  is a CTC; while  $b$  is not closed, but appears in more than one place at a given time.

This flat example is a toy model; the popular interest in CTCs is because of the possibility that CTCs lying in an otherwise almost flat space may already exist in the universe or could be created artificially. Realistic CTCs would be related to black holes or similar structures, which an object could enter at one time and exit at an earlier time (as judged by distant observers). Whatever the cause of the CTCs, evolution of a system on manifolds which possess them does not follow the familiar rules of classical physics as the following examples will demonstrate.



On manifolds with CTCs, apparently well-defined systems can have non-deterministic evolution [14]. A simple macroscopic model (see figure 4.3) demonstrates the effect.

A CTC is formed by removing two balls from 3D space (or disks from the plane) and identifying points on the boundary; this creates an ordinary wormhole. If opposing points on the two spherical (circular) boundaries are identified, but with a jump backwards in time from  $A'$  to  $A$ , then there exist closed time-like curves from  $A'$  to  $A$ , and back to  $A'$  for example. A billiard ball could pass between the mouths of the wormhole without being affected (the dotted line in figure 4.3). However, it could also be knocked into one mouth, reappear at an earlier time at the other mouth, and then hit itself - causing the first collision (the solid line in figure 4.3).

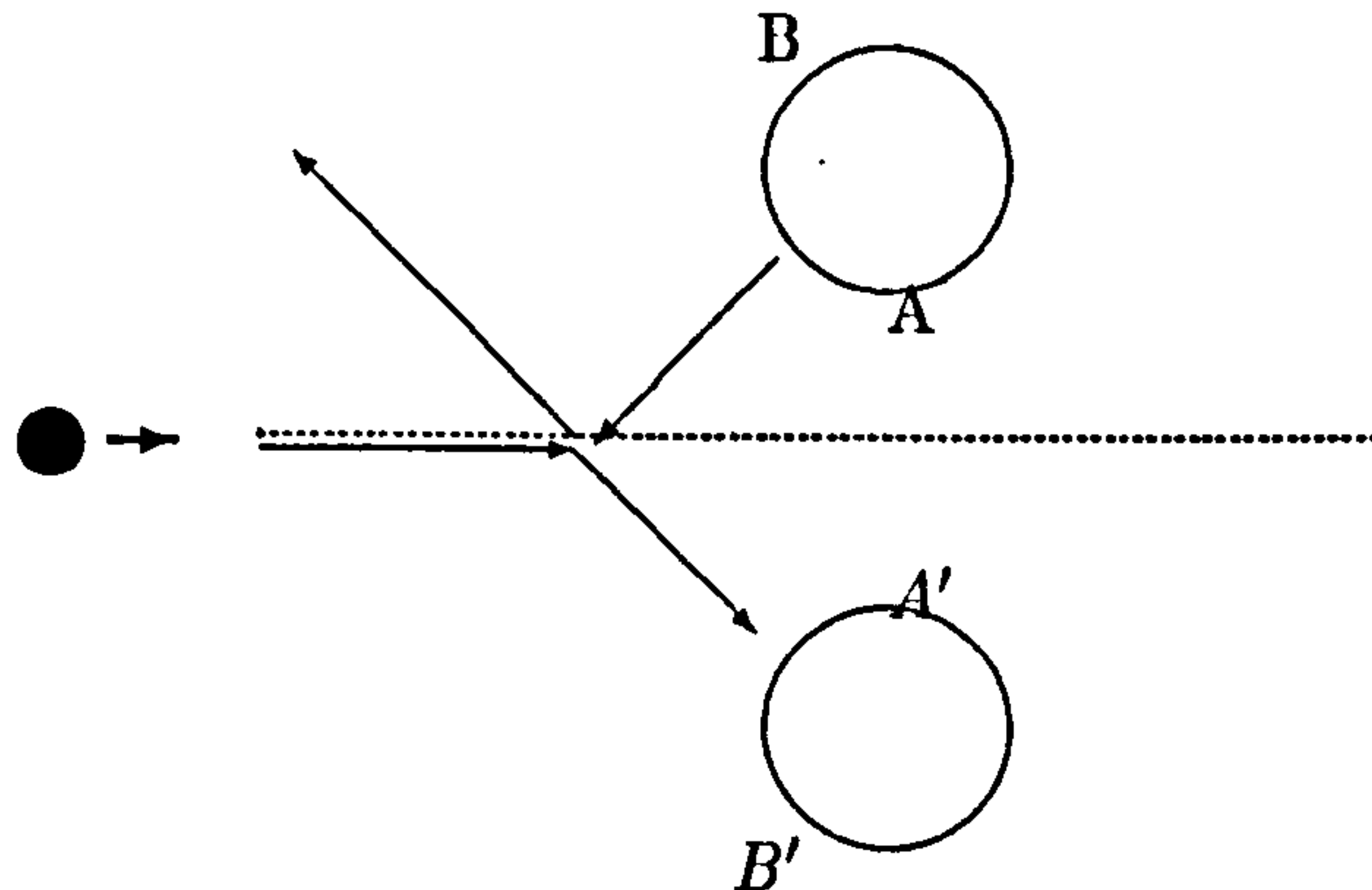


Figure 4.3: Multiple possible trajectories in a spacetime with CTCs. The ball travelling from the left may be hit by itself into one mouth of the wormhole, to emerge at an earlier time to cause the impact.

Therefore, even if all possible information about a system is given at time  $t = 0$ , the subsequent evolution still cannot be uniquely determined. This macroscopic example shows that the indeterminacy is not due to limitations on measurement but arises rather as a *consequence of the topology* of the time-like curves (the

causal structure ).

Manifolds with CTCs are usually ruled out for two reasons:

1. They are unphysical - we do not experience time travel; this however is no reason to rule them out as internal features of an elementary particle.
2. The evolution is not only non-deterministic but can also be inconsistent. Some authors postulate the existence of a consistency principle; however, Carlini *et al* [11] analyse the billiard ball example and conclude that the principle of least action eliminates the inconsistent trajectories, without having to make any additional assumptions. If this result can be extended to a general principle then the objection to CTCs disappears. The work described in this thesis assumes that only consistent solutions are physically significant (references [26, 27] also discuss the consistency issue).

It should be emphasised that the indeterminism<sup>4</sup> arises here because in a space with CTCs there can be a range of possible solutions. The existence of many solutions is essential for the work which follows.

The simple model given above was of a material object moving in a predetermined background. In general relativity we know that the object itself must distort spacetime to some degree. Although this could be an extremely small perturbation it does imply that since the position of the billiard ball cannot be predicted then neither can the metric. In general relativity, the metric would only be well defined if sufficient constraints were imposed on the problem. In this example, the initial position and velocity of the ball would have been sufficient in the absence of CTCs, *in a spacetime with CTCs they are no longer adequate.*

The features of the model which give rise to a multiplicity of possible evolutions are the CTCs together with a classical self-interacting object - the billiard ball. If the balls could not interact their trajectories would pass through each other (like geodesics which cross - which is unremarkable) and there would be only



one trajectory. Friedman and others [14] show that a quantum field can be defined unambiguously on such spacetimes but they suggest that any interacting classical particles *or fields* would exhibit a multiplicity of possible trajectories. It follows that a gravitational wave could produce the same effects, although the interaction would be attractive, rather than repulsive as in the billiard ball example.

When solving problems in general relativity which entail describing a 4-manifold it is common practice to define the metric on a spacelike slice and then calculate the evolution with time to solve for the entire manifold. However, this commonly adopted method assumes that the entire manifold can be decomposed into  $\mathbb{R} \otimes M$  - where  $M$  is any 3 manifold. This need not be the case. The global decomposition of spacetime into evolving Cauchy surfaces is only possible in spacetimes with global hyperbolicity[18, chapter 1]. It cannot be stressed too strongly that the mathematical structure of general relativity makes *no* such requirement!

In special relativity there can be no communication between spacelike separated points, because in some reference frame that would appear as cause preceding effect - in other words, a breakdown in causality. Causality is not a global feature of Einstein's theory of general relativity, but it is certainly a local feature since there always exist local coordinate systems that are approximately Lorentzian, where special relativity is valid. The non-local characteristic of quantum mechanics (as evidenced by the EPR experiments) is however, a feature of spacetimes with wormholes. The throat of the wormhole connecting otherwise distant regions of spacetime. A clear and simple example is given in[20] which is consistent with an EPR experiment. It does not seem feasible to extend the simple treatment of [20] to provide a general explanation of all non-local effects in quantum mechanics.

Solutions of Einstein's equations are known which have CTCs. The simplest is in a flat space! If we have a Lorentzian metric but identify  $t = t_1$  with  $t = 0$  the structure is rather like a cylinder with the loop not in a space direction but a time direction. A well known non-trivial example is the analytic extension of the Kerr



metric. The Kerr metric is believed to be the unique as the metric exterior to a massive body with angular momentum (of which the Schwarzschild solution is a special case) - it is therefore of real physical importance. Of course, the analytic extensions are mathematical constructions which may or may not have physical relevance. The extensions of the Kerr metric have CTCs but nothing strange happens because, as solutions, they describe the structure of spacetime, but nothing is traversing the structure. By combining the extended Kerr metric with a classical self-interacting object, such as a billiard ball, multiple consistent trajectories are conceivable.

It must be stressed that to allow CTCs requires neither a modification of, nor an addition to, the equations of general relativity. It has often been argued that CTCs are unphysical and it has been conjectured that they are prevented by a chronology protection mechanism [17]. No such mechanism has been established. By contrast, the use of CTCs merely exploits possibilities offered by standard general relativity. Historically, other theories have been formulated for one purpose and then used in regimes far removed from the original problem that the theory addressed. This is the economical way to proceed; if it is possible to explain a phenomenon with existing theories then that must always be preferred over the invention of a new theory.

## 4.4 General Relativity and the Conflict with Quantum Mechanics

General relativity and quantum mechanics are widely believed to be incompatible theories because the curvature of the spacetime manifold is regarded as a definite, deterministic property related to the density of energy and momentum by Einstein's equations, yet in quantum mechanics the same energy-momentum density cannot be known exactly at each point and only the probability of a measurable result can be predicted [32]. By understanding these objections it is possible to see how they

are circumvented in this work. The objections are seen to be either misconceptions (about general relativity) or are avoidable.

In the past, progress in fundamental physics has been achieved by exploring the limitations and inconsistencies of prevalent theories. In this way Maxwell's equations and Newtonian mechanics were reconciled by special relativity; special relativity and gravitation by general relativity; atomic structure and Maxwell's equations by quantum theory. Today the predictions of total gravitational collapse are explored because they show an inconsistency in general relativity. Similarly, the interface between general relativity and quantum mechanics needs to be reconciled. Ironically many people were surprised that it was Newtonian mechanics rather than Maxwell's equations that needed to be modified. Today the general view is that general relativity will give way to a quantum theory of one kind or another - in this work an alternative view is offered.

#### 4.4.1 The Electromagnetic Field Is Quantised

It is an empirical fact that electromagnetic fields are quantised. An argument due to Bohr and Rosenfeld (see [12, page 357] for a translation) concerning the definition and measurement of electromagnetic fields is frequently cited to show that electromagnetic fields *must* be quantised. This interpretation of their paper is refuted by Rosenfeld himself (see [34] which is reprinted in [12, page 443]). One reason that Rosenfeld gives for challenging the misinterpretation of their paper is that

a similar, and equally fallacious argument, is sometimes adduced in support of the alleged necessity of quantising the gravitational field.

I am not aware of any proof that quantisation of the electromagnetic field is a logical necessity.



#### 4.4.2 The Need for Quantum Gravity

An article by Page and Geilker [32] is frequently cited to demonstrate the need for a quantum theory of gravitation. The second part of the paper dealt with the many worlds interpretation of quantum mechanics and included an account of a short experiment which was the subject of scathing response from Ballentine [3] “A less surprising experimental result has seldom, if ever, been published” The first part of their article was also flawed, although it did hit the core of the problem of reconciling quantum mechanics and general relativity. Page and Geilker considered Einstein’s field equations in the presence of matter:

$$G_{\mu\nu} = T_{\mu\nu} \quad (4.5)$$

The left-hand-side is well-defined, continuous, smooth and evolves deterministically - it is a classical quantity. By contrast, for matter obeying the laws of quantum mechanics the right-hand-side cannot be well defined because that would imply a value for momentum, energy and all components of angular momentum to be defined precisely at each point of the manifold.

Page and Geilker mistakenly interpret the wavefunction  $\Psi$  as a physical field in space which collapses upon measurement. This view is mistaken [2, 3, 4, 5].  $\Psi(\mathbf{x})$  gives the probability of a particle being found at  $x$ ; it is a function of configuration space which is only isomorphic to a function of real space for the special case of a single particle. By the argument of Page and Geilker, the wavefunction collapses upon measurement from the average value of any parameter to an eigenvalue. This introduces a discontinuity which cannot satisfy  $\nabla G = 0$ , which is a fundamental property of the Einstein tensor. To use their interpretation of the wavefunction they rewrite 4.5 as

$$G_{\mu\nu} = \langle \Psi, T_{\mu\nu} \Psi \rangle \quad (4.6)$$

The curvature of spacetime is related, not to an actual value of energy, momentum *etc.* but, to the average value. Thus for an electron with spin-up in the  $y$ -direction,



the expected value of the  $x$ -spin would be zero and the spacetime curvature would be the same as if the  $x$ -component of spin was zero. So the angular momentum, as defined by the curvature of spacetime, would have a well-defined value of zero until an  $x$ -spin measurement was made, when it would change (presumably instantaneously) to  $\frac{1}{2}$  or  $-\frac{1}{2}$ . The advantage of their interpretation is that the correct results are obtained for an ensemble (and hence a macroscopic body).

It cannot be stressed too strongly that  $\Psi(\mathbf{x})$  is not a physical entity, it does not describe a particle that is spread out. It is a probability function which can give the probability of finding an entire (point) particle at the point  $x$ ; the importance of this (correct) interpretation is stressed by Ballentine [5, 2]. A helpful analogy is a model of a dart hitting a dartboard. The dart is always in one place and only one place. When it hits the dartboard it will always end up in one place only. However, because it is a stochastic process, a description of where it might hit the board can only give a probability density for different positions on the board. It would clearly be ridiculous to do physical calculations based on 0.33 of a dart scoring 20. Equally it is ridiculous to suggest that the dart is at the average position (the bulls eye perhaps!).

In physics it is frequently either convenient, or a good approximation to relate one physical quantity to the average of another one for example a macroscopic quantity, such as pressure, to the average of (microscopic) molecular momenta. However, I cannot envisage a precise fundamental physical parameter, in this case the curvature of space, which is given *exactly* in terms of the *average of possible* values of another one. Therefore equation 4.6 is not tenable as an equation for the actual exact curvature due to a single particle; it could be averaged, but not as Page and Geilker did by averaging one side of the equation only! A modified form of equation 4.6 could be used for the average value of the Einstein tensor:

$$\langle G_{\mu\nu} \rangle = \langle \Psi, T_{\mu\nu} \Psi \rangle, \quad (4.7)$$

where the average is over an ensemble of possible outcomes, but it cannot reasonably

give the *exact* value of the curvature due to a *single* particle as an *average* of all the things that might happen!

Page and Geilker have succeeded in highlighting where the problem lies, even if their analysis is flawed. The left hand side of equation 4.5 obeys classical laws while quantum mechanics describes the right hand side and so it cannot have a well-defined value. Indeed, if equation 4.5 were correct (and if  $G_{\mu\nu}$  were well-defined) it would offer a way to complete quantum mechanics by measuring the gravitational field of a single particle in an asymptotic region, finding the energy momentum, centre of mass, and all components of angular momentum to arbitrary high accuracy [30, chapter 19]. Experimentally this may be unachievable, but the fact is that if  $g_{\mu\nu}$  is well-defined in the asymptotically flat regions of space then it can be used to define all of a particle's parameters - in contradiction not only to the accepted view of quantum mechanics but also with experimental facts.

### The Region of Conflict

It is frequently quoted that quantum gravity becomes a significant factor only on the scale of the Planck length - this is misleading. The conflict apparent in equation 4.5 arises already at *atomic* dimensions,  $10^{-10}m$ , as quantum effects become significant. The references to the Planck length are to the distances where gravitational effects have similar energy to other forces, it is used as a measure of energy. In principle there is a conflict even for billiard balls since general relativity gives a way of defining position and momentum simultaneously to any degree of accuracy - contrary to the quantum mechanical predictions even for massive bodies.

### Status of Bohm's Theory

The preceding comments would presumably not be a problem for advocates of Bohm's theory where particles do have definite measurable parameters at all times, but are guided by a non-local quantum field. In Bohm's theory the asymptotic

form of the metric would be well defined. The cost is that the particle moves under the influence of the quantum field which does not conserve momentum *etc.* Bohm's theory therefore offers a self-consistent way of assigning a definite value to position, momentum, *etc.* and hence to the metric, in full accordance with quantum theory.



## Chapter 5

# Geons and Measurements

### 5.1 Geons

Since the advent of the theory of general relativity attempts have been made to describe elementary particles as distortions of space and time, usually as structures with non-trivial topology. Einstein[13] sought to unify fields and particles by having a field description of particles. Ironically his attempts sought spherically symmetric solutions for particles, motivated by the small number of known particles at that time - the proton and electron!

Wheeler [31] used the name geon to describe an object composed of gravitational and electromagnetic fields, held together by its own energy, and showed how wormhole structures could explain the appearance of charges and mass from source-free field equations. Interest in wormhole topologies has continued to this day, although the emphasis is no longer on them as models of elementary particles.

Brill and Hartle [9] describe a geon in which orbiting electromagnetic or gravitational waves cause sufficient curvature of the background metric that they are bound by their own energy. They find that the size,  $r$ , and the mass,  $M$ , are related by:

$$r = \frac{9G}{4c^2}M \quad (5.1)$$

The object has approximately spherical symmetry.

Sorkin [15] has constructed geons with unorientable manifolds which could explain why only one type of charge exists (electric but not magnetic).

Geons fell into disrepute because no singularity free examples could be constructed. There were increasingly tight constraints imposed by singularity theorems which predicted the collapse to singularities of gravitational structures. The wormhole topologies would all collapse to a singularity in the future, and would also collapse to a singularity before they could be traversed [30, p 838]. There was also no indication of how they could be quantised - wormholes, for example, can be constructed for any value of mass and for most values of charge and angular momentum. They are classical objects which did not seem to fit as descriptions of elementary particles governed by quantum mechanics.

It is strange that the idea of geons is no longer taken seriously since, as Wheeler showed, they can potentially explain what an elementary particle is, and how charge and mass arise.

## 5.2 Measuring a Geon

Measurement of position is not a trivial matter. The remainder of this chapter examines the measurement of position, both from a practical point of view and in a theoretical way. A definition of position is given which is applicable to a range of objects from astronomical to the microscopic, from the tangible to the intangible, to both classical objects and to quantum particles. In each case a practical definition is sought that has reasonable properties and which is consistent with the other cases.

The gravitational field in the asymptotic region also gives a measure of position (as well as other properties) which is applicable to astronomical objects. The very existence of such a gravitational field has fundamental ramifications for the reconciliation of quantum mechanics and general relativity.

### 5.2.1 The Position of a Geon

First we consider measuring the position of an *ordinary* object such as a billiard ball. As the discussion evolves it will be clear that the techniques can be extended to cover an enormous range of objects but there are some limits to the applicability.

1. **Touching:** A common, intuitive notion of position is to *feel where it is*. This means making contact with a reference object (the hand, callipers, stick, *etc.*). The type of contact is not important.
2. **Confining:** The object is in a box, confined by it in at least one direction. Clearly the walls of the box need to be an effective barrier to the object, and it must be known which side of the walls the object is! Even a single (infinite) wall can give some measure of position *eg.*  $x > 0$  or  $x < 0$ .
3. **Seeing:** Light from any source shines on the object and the direction of the reflected light is detected. The light is assumed to travel in a straight line and the angle at which it reaches the detector determines the angular position, but not the distance.
4. **occulting:** The angular position is determined when the object blocks the light from a known source. The angular position is then equated with that of the known light source.

Of these common procedures, touching is the most limited; microscopic objects would be influenced by the interaction in a possibly unpredictable way, and for distant objects it is not achievable. Seeing has clear astronomical usefulness for any bright object and occulting for most dull objects.

#### Position defined by curvature of spacetime

Black holes are a challenging exception. They do not emit light - so seeing is not an option. Light from behind them does not necessarily travel in a straight line



because of the distortion of space giving gravitational lensing effects. Touching, by sending a probe and monitoring the probe's position would give the strange result that the black hole was an infinite distance away, because a distant observer would never see the probe go beyond the event horizon. Clearly confinement is not feasible. The technique actually employed is to measure the position of a visible associated (orbiting) body. In effect, this is measuring the gravitational field of the black hole using an external test particle (the companion star!).

In effect the position is inferred from the gravitational field using the Schwarzschild solution which provides a chart from real curved spacetime to flat polar coordinates. With an appropriate choice of coordinates, the centre of the black hole is defined to be at  $r = 0$ , where  $r$  is a radial coordinate in a system of coordinates which gives the closest approximation to a Schwarzschild metric. To use this technique it is necessary to be able to measure the gravitational field and to have an asymptotically flat background so that the real space and the map to flat space become asymptotically identical. The same method can be used for an arbitrary source where the metric at a large distance away is given by:

$$\begin{aligned}
 ds^2 = & - \left( 1 - \frac{2M}{r} + \frac{2M^2}{r^2} + O\left(\frac{1}{r^3}\right) \right) dt^2 \\
 & - \left( 4\epsilon_{jkl} S^k \frac{x^l}{r^3} + O\left(\frac{1}{r^3}\right) \right) dt dx^j \\
 & + \left( 1 + \frac{2M}{r} + \frac{3M^2}{2r^2} + O\left(\frac{1}{r^3}\right) \right) dx^k dx^k
 \end{aligned} \tag{5.2}$$

in a coordinate system that has the centre of mass at  $r = 0$ . The mass is given by  $M$  and the angular momentum by  $S^k$ . To define the mass *etc.* from the gravitational field an invariant expression is required, which is valid in any reference frame. This can be done. In an arbitrary frame a quantity  $H^{\mu\alpha\nu\beta}$  first needs to be defined:

$$H^{\mu\alpha\nu\beta} \equiv -(\bar{h}^{\mu\nu}\eta^{\alpha\beta} + \eta^{\mu\nu}\bar{h}^{\alpha\beta} - \bar{h}^{\alpha\nu}\eta^{\mu\beta} - \bar{h}^{\mu\beta}\eta^{\alpha\nu}) \tag{5.3}$$

where  $\bar{h}^{\mu\nu} \equiv h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h^\alpha_\alpha$  and  $h_{\mu\nu}$  is the departure of the metric  $g_{\mu\nu}$  from the Lorentz metric  $\eta_{\mu\nu}$ . In terms of surface integrals of  $H^{\mu\alpha\nu\beta}$  evaluated at a large

distance from the source we can define:

$$P^\mu = \frac{1}{16\pi} \oint_S H_{,\alpha}^{\mu\alpha 0j} d^2 S_j \quad (5.4)$$

$$P^0 = \frac{1}{16\pi} \oint_S (g_{jk,k} - g_{kk,j}) d^2 S_j \quad (5.5)$$

$$J^{\mu\nu} = \frac{1}{16\pi} \oint_S (x^\mu H_{,\alpha}^{\nu\alpha 0j} - x^\nu H_{,\alpha}^{\mu\alpha 0j}) + H^{\mu j 0\nu} - H^{\nu j 0\mu} d^2 S_j \quad (5.6)$$

$$M = \sqrt{(-P^\mu P_\mu)} \quad (5.7)$$

$$Y^\mu = -J^{\mu\nu} P_\nu / M^2 \quad (5.8)$$

$$S^\rho = \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} (J^{\mu\nu} - Y^\mu P^\nu + Y^\nu P^\mu) P^\sigma / M, \quad (5.9)$$

where  $Y^\mu$  is the displacement of the centre of mass from the origin of the chosen coordinate system. The intrinsic angular momentum is  $S^\rho$  - as opposed to the angular momentum about the origin of coordinates,  $J^{\mu\nu}$ .

The information in the asymptotic form of the metric is of fundamental theoretical significance (and occasional astronomical value). The theoretical significance is that (even if it cannot be measured) the very existence of a well-defined asymptotically flat metric means that the object has well-defined energy, momentum, angular momentum and position, in contrast to the situation in quantum mechanics (Bohm's theory excepted). Despite the fundamental significance for the theory of quantum physics, it has no experimental use for an elementary particle (or any microscopic object) because it is not feasible to measure such small gravitational fields. If geons are postulated as models for elementary particles their position cannot be determined, in practice, from the gravitational field. An alternative, practical, definition must be sought.

## Confinement

The notion of confinement as a way of defining the position of an elementary particle is experimentally reasonable. Like touching, it may influence the particle, but the substantive nature of the barrier ensures that, as a measure of position, the

confinement gives a meaningful result. This definition has the advantage of giving a measure even if the metric of spacetime near the particle differs substantially from the flat metric; it even applies if the topological structure of spacetime is nontrivial. Compare this with definitions in terms of geodesics which give unacceptable answers for black holes and any other objects with substantial deviations from a flat topologically trivial metric.

Confinement as a definition of (and as a means of measuring) position is practical for elementary particles; it is the *only* practical way of measuring the position of a geon. It is suggested that, in practice, all measures of position (or at least the non-destructive ones) rely on confinement.

### 5.2.2 Localisability

A fundamental property of a particle, without which an object would not be called a particle, is that it will always be found in one place and only one place. This is slightly different from saying that it will always be in one place. In the quantum arena the first statement is true and meaningful whereas the second is widely believed to be untrue.

### 5.2.3 Measuring Other Parameters

Other properties of a particle can be measured by using a filter followed by a position measurement. The localisability of a particle will ensure that a single result is achieved (see figure 7.2). By a filter we mean an idealised experimental apparatus which splits up incoming particles according to some property (*eg.*  $x$ -component of angular momentum) in such a way that subsequent filters, of the same type, will cause *no further* splitting of the beam. Thus the filter not only acts according to the state of the particles but also prepares them in a known state.



## Chapter 6

# A Gravitational Explanation of Quantum Mechanics

In this chapter a model for elementary particles is described based on the novel concept of a 4-geon. The model is constructed in the framework of classical general relativity, but is consistent with quantum mechanics and all known properties of elementary particles. The description is not detailed to the extent of giving an exact solution of Einstein's equations - all the advice I received strongly cautioned against trying to find exact solutions as they are notoriously difficult to find. Consequently this work rests on certain, reasonable propositions about the existence and form of solutions. It will remain speculative until such solutions are found. A discussion comparing the speculative aspects of this work with alternatives such as string theory and quantum gravity theories is in chapter 9.

### 6.1 Using CTCs

In section 4.3 it was shown how billiard balls moving in a predefined spacetime with CTCs could have more than one consistent trajectory. The paradoxes, and other interesting features, in science-fiction stories about time travel arise from the same

two requirements; the existence of CTCs which allow the time travel *and* the ability of an object (usually the time traveller) to interact with an earlier self (directly or indirectly).

The gravitational field can be regarded as a classical field and weak gravitational fields can be regarded as perturbations of a pre-existing spacetime. If the linear approximation is valid then there can be no possibility of self-interaction. However, even weak fields fail to satisfy the conditions for a linear approximation to be valid if the frequency is high [23]. One can therefore envisage a combination of a Kerr background together with high frequency ripples. This would be a solution of the vacuum equations where the ripples at least had more than one possible evolution. This is not proposed as a model of an elementary particle but demonstrates some of the ingredients for a solution. The distinction between background and ripple is easy to work with, but unnatural, and may not give the variety of possible evolutions needed to account for quantum phenomena. Unfortunately, it is difficult enough to imagine 3-manifolds, harder still simple 4-manifolds, while attempts to visualise convoluted 4-manifolds with non-trivial causal structure is prohibitive. The purpose of this work is to draw as many conclusions as possible *without* having to construct an explicit structure - indeed the preceding work on the foundations of quantum mechanics shows just how much can be achieved without knowing exactly what an elementary particle is. You can even derive Schrödinger's equation without knowing what the  $\Psi$  is (it could be a probability, a particle density, charge density or a wave in space) all would give the same equation if a unitary representation and Galilean invariance were adopted.

## 6.2 The 4-Geon

The present analysis is based upon a model of an elementary particle as a distortion of spacetime (a four dimensional semi-Riemannian manifold with non-trivial topology). The manifold includes both the particle and the background metric, and

being four dimensional without a *global* time coordinate, the particle and its evolution are inseparable - they are both described by the 4-manifold. We now express the properties, which we require of a particle, in the language of manifolds.

**Axiom 1 (Asymptotic flatness)** *Far away from the particle spacetime is topologically trivial and asymptotically flat with an approximately Lorentzian metric.*

In mathematical terms - spacetime is a 4-manifold,  $\mathcal{M}$ , there exists a 4-manifold  $K$ , such that  $\mathcal{M}/K$  is diffeomorphic to  $\mathbb{R}^4/(\mathbb{B}^3 \times \mathbb{R})$  and the metric on  $\mathcal{M}/K$  is asymptotically Lorentzian.<sup>1</sup>  $K$  or  $(\mathbb{B}^3 \times \mathbb{R})$  can be regarded as the world-tube within which the 'particle' is considered to exist.

This axiom formally states the fact that we experience an approximately Lorentzian spacetime, and that if space and time are strongly distorted and convoluted to form a particle then that region can be localised. (It may be noted that asymptotic flatness is not a reasonable property to require for a quark because it cannot be isolated [there is no evidence of an isolated quark embedded in a flat spacetime] therefore the present work cannot be applied automatically to an isolated quark.)

The position of a distortion of spacetime is not a trivial concept - it implies a mapping from the 4-manifold, which is both the particle and the background spacetime, onto the flat spacetime used within the laboratory. There is in general no such map that can be defined globally, yet a local map obviously cannot relate the relative positions of distant objects. This axiom gives a practical definition of the position of a particle - it is the region where the non-trivial topology resides. Any experimental arrangement which confines (with barriers of some sort) the  $\mathbb{B}^3$  region of non-trivial topology, defines the position of the particle. From this axiom, the region outside the barrier is topologically trivial and therefore *does* admit global coordinates.

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<sup>1</sup> $\mathbb{B}^3$  is a solid sphere



Using the asymptotic flatness axiom it is now possible to define what is meant by a particle-like solution:

**Axiom 2 (Particle-like)** *In any volume of 3-space an experiment to determine the presence of the particle will yield a true or a false value only.*

This is consistent with the non-relativistic indivisibility of the particle. By contrast, a gravitational wave may be generally a diffuse object with a density in different regions of space which can take on a continuous range of values. An object which did not satisfy this axiom (at least in the non-relativistic approximation) would not be considered to be a particle. The axiom is clearly satisfied by classical particles and, because it refers only to the result of a position measurement, it conforms also with a quantum mechanical description of a particle.

The particle-like axiom requires the property of asymptotic flatness, defined above, to give meaning to a 3-space. The three space is defined in the global asymptotically flat, topologically trivial region,  $\mathcal{M}/K$ , which is diffeomorphic to  $\mathbb{R}^4/(\mathbb{B}^3 \times \mathbb{R})$  as defined above.

We are now able to state the required properties of a 4-geon.

**Conjecture 1 (4-Geon)** *A particle is a semi-Riemannian spacetime manifold,  $\mathcal{M}$ , which is a solution of Einstein's equations of general relativity. The manifold is topologically non-trivial, with a non-trivial causal structure, and is asymptotically flat and particle-like (Axiom 2).*

It would be very appealing if  $\mathcal{M}$  was a solution of the vacuum equations[13], but for the arguments that follow this is not essential; unspecified non-gravitational sources could be part of the structure. The assumed existence of CTCs *as an integral part of the structure* (rather than as a passive feature of the background topology) is an essential feature of the manifold; when these conditions exist additional boundary conditions may be required to define the manifold[14]. This aspect of the structure provides a causal link between measurement apparatus and state preparation,

permitting *both* to form part of the boundary conditions which constrain the field equations.

The axioms formally state conditions that any description of a particle must reasonably be expected to satisfy. In contrast, the 4-geon (Conjecture 1 above) is novel and speculative since it is not known whether such solutions exist - either to the vacuum or the full field equations of general relativity; however, *there no reason to suppose that they cannot exist*. It will be shown that this single speculative element not only yields quantum logic, but is sufficient to derive the equations of quantum mechanics and in doing so reconciles general relativity with quantum mechanics. Although this work exploits novel and unproven structures (CTCs) in general relativity it requires neither a modification, nor any addition to Einstein's equations; the number of spacetime dimensions remains  $3 + 1$ . The work does not require extraneous particle fields (as used in conventional quantum field theory), nor does it impose a quantum field of unknown origin (as does Bohm's theory). In short, this single conjecture is sufficient to unify particle and field descriptions of Nature, and quantum mechanics with general relativity.

### 6.3 Measurement of a 4-geon

The role of both the measurement and state preparation in defining the 4-geon is crucial. It is self-evident that state preparation sets boundary conditions. Whether we regard a particle as a classical billiard ball, a quantum of a quantum field, or a classical field, the state preparation limits the possibilities; it restricts the possible solutions to those consistent with the apparatus. Systems with slits, collimators and shutters provide obvious boundary conditions which any solution must satisfy. For a geon, or a 4-geon, a barrier is a region which the topologically non-trivial region cannot traverse. Such barriers can be used to form slits and collimators *etc.* and they obviously restrict the space of possible solutions. We state this formally as an axiom:



**Axiom 3 (State preparation)** *The state preparation sets boundary conditions for the solutions to the field equations.*

The exact nature of these boundary conditions, and whether they can always be equated with physical barriers such as collimators, is irrelevant to the analysis that follows.

Consider now an apparatus associated with a measurement, which is in many respects similar to that involved in a state preparation; arrangements of slits, barriers and collimators are common features of the measurement apparatus. They are constructed from barriers which cannot be traversed by the non-trivial topology, which is the particle.

We take as a paradigm for a position measurement that barriers divide space into regions which are then probed (in any manner) to ascertain the existence, or otherwise, of the particle in a region. The particle-like axiom and the asymptotic flatness axiom assures us that the topologically non-trivial region can be confined but not split.

We take the view of Holland [20] that most measurements can be reduced to position measurements. A sequence of shutters and collimators and filters (*eg.* such as used in a Stern-Gerlach apparatus) determines the state preparation, while a very similar system of shutters *etc.*, resulting in confinement to one of a number of regions and subsequent detection, acts as a measurement.

For a classical object there is no causal connection that could allow the measurement conditions to influence the evolution. If the state preparation was insufficient to uniquely specify the trajectory there would be a statistical distribution of possible initial states, each of which would evolve deterministically. By contrast, on a spacetime with CTCs *extra* conditions are required for a unique deterministic evolution [14]. With a particle modelled as a 4-geon however, there *would* be a causal link allowing the measurement conditions to contribute to the definition of the 4-manifold. A 4-geon is a 4-dimensional spacetime manifold which satisfies the



boundary conditions set by *both* the state preparation *and* the measurement. This justifies a further axiom:

**Axiom 4 (Measurement process)** *The measurement process sets boundary conditions for the 4-geon which are not necessarily redundant, in the sense that they contribute to the definition of the 4-manifold.*

This axiom is inevitable if the particle contains CTCs, because the state preparation and the measurement conditions can no longer be distinguished by causal arguments.

**Axiom 5 (Exclusive experiments)** *Some pairs of measurements are mutually exclusive in the sense that they cannot be made simultaneously.*

This axiom expresses an established experimental fact - see[36, Chapter 7]. The famous examples of two such complementary variables are  $x$ -position and  $x$ -momentum. The  $x$  and  $y$  components of spin form another pair of complementary variables with a very simple logical structure. That measurements cannot be made simultaneously is still consistent with classical physics; objects would have a precise position and momentum, but we could only measure one property or the other. Quantum mechanics goes much further and asserts that a particle cannot *even* possess precise values of both properties simultaneously. The present work is unique in explaining why an inability to make simultaneous measurements should lead to incompatible observables in the quantum mechanical sense.

## 6.4 Propositions and 4-Manifolds

We now consider the semi-Riemannian manifolds,  $M$ , that could satisfy the different boundary conditions imposed by state preparation and measurement: Let  $\mathcal{M} \equiv \{M\}$  denote the set of 4-manifolds consistent with the state preparation conditions; there is no reason to suppose that a  $M$  is unique. The inability to define  $M$  uniquely will result in a *classical* distribution of measurement results.

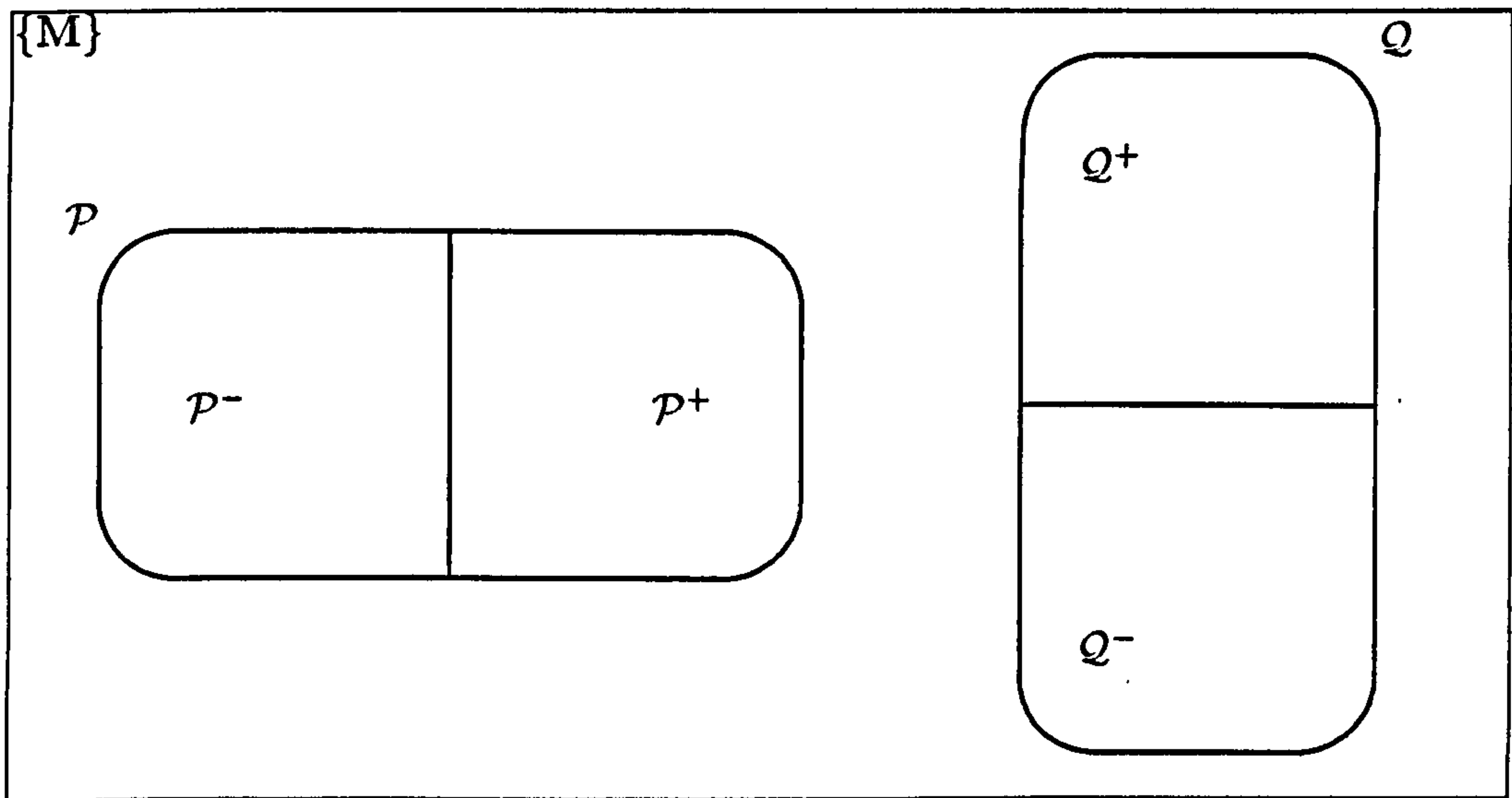


Figure 6.1: Sets of 4-manifolds consistent with both state preparation and the boundary conditions imposed by different measurement conditions.

The 4-manifold describes both the particle and its evolution; for a 4-geon they are inseparable. Consequently, the terms *initial* and *evolution* need to be used with great care. Although valid in the asymptotically flat region (and hence to any observer), they cannot be extended throughout the manifold. Preparation *followed by* measurement is also a concept valid only in the asymptotic region: *within the particle causal structure breaks down*.

Consider first the case of the classical 3-geon. Each  $M$  corresponds to an evolving 3-manifold  $M^3$ . Each  $M^3$  will evolve deterministically in a way determined uniquely by the field equations, the initial condition  $M^3(t_0)$ , and the extrinsic curvature (the distribution of  $M^3(t_0)$  determines the distribution of  $M^3(t > t_0)$ ). If the geon is particle-like, then any experiment that depends upon a position measurement will give a result for each  $M^3$  at any time. Consequently, the boundary conditions imposed by measurements are necessarily compatible with any 3-geon

that satisfies the particle-like proposition; in other words they are redundant.

By contrast, the 4-geon with CTCs as part of its structure cannot be decomposed into a three manifold and a time variable. It is known that further boundary conditions need not be redundant in a spacetime which admits CTCs[14]. In principle, the measurement apparatus itself can provide additional boundary conditions.

Consider measurements  $P, Q$  for which the boundary conditions cannot be simultaneously applied. They could be the  $x$ -component of spin and  $y$ -component of spin, or  $x$ -position and  $x$ -momentum; for simplicity we will consider two-valued measurements (*eg.* spin up or down for a spin-half particle or  $x$ -position  $> 0$  and  $x$ -momentum  $> 0$ ). We will denote the result that “*the state has a +ve  $P$  value*” by  $P^+$ , ( $P^-, Q^+, Q^-$ , are defined similarly). As a proposition  $P^+$  is clearly the complement of  $P^-$ ; if  $P^+$  is true then  $P^-$  is false and *vice versa*, and similarly for  $Q^+$  and  $Q^-$ .

As before, let  $\mathcal{M} \equiv \{M\}$  denote the set of 4-manifolds consistent with the state preparation. The measurements define subsets of  $\{M\}$ ; we denote by  $\mathcal{P}$  those manifolds consistent with the state preparation *and* the boundary conditions imposed by a  $P$ -measurement.  $\mathcal{P}$  is clearly the disjoint union of  $\mathcal{P}^+$  and  $\mathcal{P}^-$  - the manifolds corresponding to  $P^+$  and  $P^-$ , respectively. Where the boundary conditions imposed by the measurement are not redundant  $\{M\}$ ,  $\mathcal{P}$  and  $\mathcal{Q}$  need not be the same (see Figure 6.1). There is a one-to-one correspondence between the sets of manifolds in the Figure and the four non-trivial propositions,  $p, q, r, s$ . However, two statements, or experimental procedures correspond to the same proposition if they cannot be distinguished by any state preparation - in other words if they give exactly the same information about each and every state. Therefore the statement that  $P$  has a value is always true by the particle-like Axiom 2, as is the statement that  $Q$  has a value; hence the subsets  $\mathcal{P}$  and  $\mathcal{Q}$  correspond to the *same* proposition  $I$  and we have the possibility:

$$\mathcal{P}^+ \neq (\mathcal{P}^+ \cap \mathcal{Q}^+) \cup (\mathcal{P}^+ \cap \mathcal{Q}^-) \quad (6.1)$$



If the boundary conditions are incompatible then  $\mathcal{P}$  and  $\mathcal{Q}$  are disjoint and the following holds (see Figure 6.1):

$$0 = (\mathcal{P}^+ \cap \mathcal{Q}^+) = (\mathcal{P}^+ \cap \mathcal{Q}^-) \neq \mathcal{P}^+ \quad (6.2)$$

Therefore, propositions about a state do not necessarily satisfy the distributive law of Boolean algebra.

## 6.5 General Relativity and Quantum Mechanics

The significance of this result (equation 6.1) is that the failure of the distributive law is synonymous with the existence of incompatible observables[7, page 126]; it is a definitive property of non-classical systems of which a system obeying the rules of quantum mechanics is an example. To obtain quantum mechanics (as represented by a projections of a Hilbert space) we need to replace the distributive law with the weaker orthomodular condition:

$$a \leq b \Rightarrow b = a \vee (b \wedge \text{NOT}(a)) \quad (6.3)$$

where  $\leq$  is a partial ordering relation which is transitive, reflexive and antisymmetric; it corresponds to set theoretic inclusion of the manifolds,  $\mathcal{A} \subseteq \mathcal{B}$ . Mackey's arguments (see [28], or [7, chapter 13] for a clear summary of the ideas) show that a modular poset is a minimum requirement for a system of propositions that is consistent with the physical requirements of making a measurement. The existence of a lattice rather than merely a poset is one consequence of this work (see section 7.2), while the use of orthomodularity rather than modularity is a technicality relevant to observables with an unbounded spectrum (see section 7.3).

For propositions,  $a$  and  $b$ , the ordering relation can only be applied if they can be evaluated together [7, chapter 13]. When  $a \leq b$  there must be a measurement apparatus which enables  $a$  and  $b$  to be measured together. Let  $\mathcal{P}$  be the subset of  $\mathcal{M}$  defined by this measurement (see Figure 6.2). Then  $\mathcal{A}^+ \subseteq \mathcal{B}^+ \subseteq \mathcal{P}$  and the

complements with respect to  $\mathcal{P}$  satisfy  $B^- \subseteq \mathcal{A}^- \subseteq \mathcal{P}$ . Clearly:

$$\mathcal{A}^+ \subseteq B^+ \Rightarrow B^+ = \mathcal{A}^+ \cup (B^+ \cap \mathcal{A}^-) \quad (6.4)$$

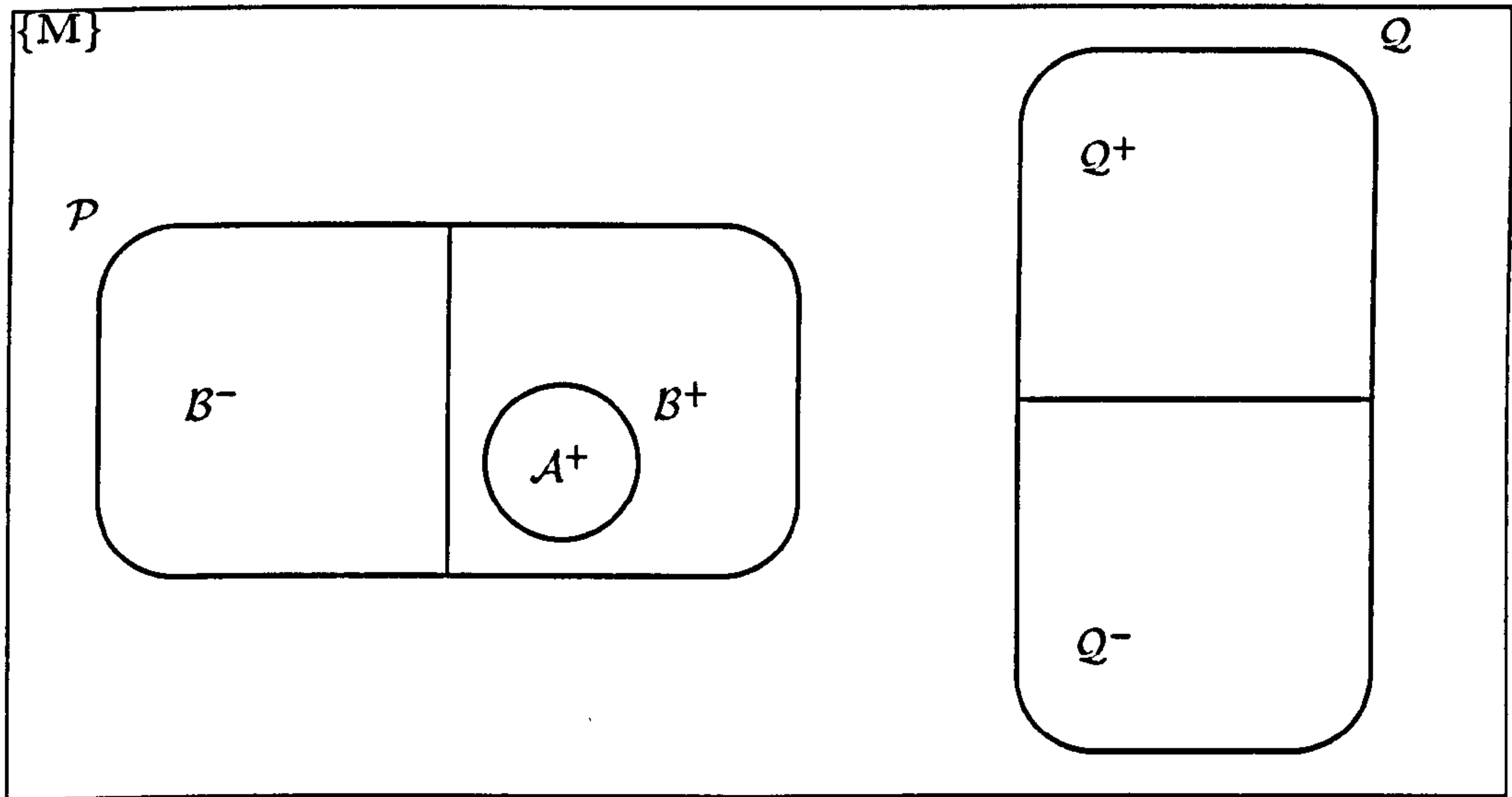


Figure 6.2: Sets of 4-manifolds illustrating the orthomodular condition for compatible propositions,  $a$  and  $b$ .

Hence the weaker orthomodularity condition is satisfied by propositions about the 4-geon manifolds. Quantum mechanics (as represented on a Hilbert space) is a realization of non-distributive proposition systems which satisfy equation 6.3, and is believed to be unique as a representation on a vector space. For a review and further references on the relation between non-distributive proposition systems, quantum mechanics and complex Hilbert spaces see [7, chapters 21,22].

## 6.6 Comparison of a 4-geon and a Classical Geon

In this section the properties of a classical geon and a geon with CTCs are compared in detail. A distinction is made between those properties which are well defined in

the theory of general relativity and those which are accessible to experiment. The two types of geon are dealt with in turn and the degree to which their properties do, or could be contrived to, agree with experimental results is discussed. Finally the two cases are directly compared.

### The Classical Geon

This is a three-dimensional space manifold which evolves in time. The manifold is uniquely defined and is asymptotically flat. The energy and momentum are well defined by the asymptotic properties of the metric; see Equations 5.4 *etc.* as given by [30] and [39]. The metric far from an arbitrary source will be asymptotically spherical (see [30] chapter 19). The position will also be given by the form of the asymptotic metric - basically it is at  $r = 0$  in the spherical coordinate system which is closest to the Schwarzschild solution.

Note that the curvature due to the geon is not compactly supported (*ie.* it extends to infinity), otherwise the geon would be massless and of zero energy. It is also vital that, although the geon may be confined in some apparatus, the asymptotic form of the metric has a contribution due to the geon which can be used to give a mathematical definition of the mass, energy momentum, angular momentum and position. In other words mass cannot be screened. A consequence of this is that the energy of the particle extends to infinity and therefore the presence of a particle cannot be equated with all its energy being localised; one must define confinement in terms of 'most' of the energy, or in terms of some other property such as the topological structure. This is reasonable since the detection of say an electron would normally be understood to mean not just that its energy had been detected but the electronness also (as characterised by the electron lepton number and the charge). In a geon model the particle properties are associated with the topological structure, which is in turn the source of the mass and energy.

The motion of a classical geon is deterministic - the evolution of the three-



manifold can be calculated using the equations of general relativity. If it left a source in a known state then the state at a distant screen/detector would be known - including a definite position, energy, momentum and every component of angular momentum (see chapter 20 of [30] for a description of how Einstein's equation determine the motion of any object, or see the chapter by Peter Havas in [22] for a review of subject). If some parameters were unknown then only a probability distribution could be given and we would have a simple hidden-variables model - a classical stochastic process.

The geon is a solution to the Einstein equations of general relativity which satisfies given boundary conditions. The state preparation will certainly impose some boundary conditions. Experimental limitations will prevent the state preparation from uniquely determining the manifold. The measurement process cannot impose additional boundary conditions without contradiction. To see this we can appeal to the rules of causality or the equations of general relativity[30]. The results of a measurement may well have a stochastic nature because several different manifolds could be consistent with the state preparation. However this uncertainty is of the classical statistical type - it has been proven not to reproduce the results of quantum mechanics (see the work of Neumann, Kochen and Specker, as described in [6]).

The classical geon behaves like a classical particle with definite properties, independent of measurements, unable to exhibit interference effects or wave-like character.

For a classical geon to exhibit interference then it must be an extended object - some interaction of position measuring instrument and the geon must be envisaged to force the extended object to collapse to a small region when position is measured. Alternatively, some construction must be employed such as Bohm's theory where the particle is localised but is guided by an extended wave-function. It is conceivable that the geon has a localised topological structure together with an extended gravitational

perturbation; the former accounts for the results of a position measurement while the latter could explain interference effects as in Bohm's theorem. However this construction fails to reproduce the non-local effects of quantum mechanics - Bohm's quantum wave-function is non-local, whereas a gravitational perturbation can only transmit information at the speed of light.

Angular momentum is another problem for a classical geon description. The angular momentum is well defined, both in magnitude and direction. Consequently, for any non-zero angular momentum, the  $z$ -component can take on a continuous range of values depending upon the orientation of the axes. Even if some unspecified mechanism forces measurements to take only the values seen in quantum mechanics, the Kochen and Specker paradox (as described in [6]) still contradicts idea that particles possess well-defined angular momentum components *independently* of the measurements that may be made.

Surprisingly, the EPR paradox (in its current interpretation using particle spins) *can* be explained by a classical geon. The observed non-locality for measurements of two particles can be interpreted as single topological structure with each particle being one mouth of a wormhole [20]. Unfortunately, this picture which seems simple for this single example, is difficult to reconcile with the different ways in which particles can be created, paired, measured and annihilated. An obvious problem is that all electrons are identical, yet EPR experiments would require electron-electron wormholes while other experiments would create electron-anti-electron pairs *etc.*

In conclusion: a classical geon behaves like a classical particle. This is a consistent interpretation. When some of the geon properties are unknown, probabilities can be assigned to the outcomes of measurements in accordance with the usual rules of statistics. Wave-particle duality, quantisation of spin-components, non-locality and the complementarity of position and momentum all pose major problems of interpretation for a classical geon structure. There is no known theory based on geons which can account for all these phenomena, although contrived constructions



can explain one or two in isolation.

## A 4-Geon

A 4-geon is a four dimensional space-time (Lorentzian) manifold which *cannot* be globally trivialised as the product of a three manifold (with +ve definite metric) and  $\mathbb{R}$  (a time coordinate). Asymptotic flatness is assumed, however, in order to conform with the classical world which we observe.

The geon is assumed to be particle-like in the sense that a position determination in any region of space gives a single value; yes or no.

This geon is *not* an evolving 3-manifold. It *cannot* be because there is no global time coordinate to give meaning to the term 'evolving'. For similar reasons the term 'initial', as referred to an initial time, cannot have the usual meaning.

The geon is a solution to the Einstein equations of general relativity which satisfies given boundary conditions. The state preparation will certainly impose some boundary conditions. As in the case of the classical geon, experimental limitations will prevent the state preparation from uniquely determining the manifold. However, in this case, *the measurement process itself may impose additional boundary conditions without contradiction*. The causal arguments applied in the case of a classical geon break down when a global time cannot be defined.

An attempt to describe the particle only in terms of the result of the state preparation will then have peculiar consequences. Some 4-manifolds compatible with the boundary conditions imposed by the state preparation will be incompatible with the the boundary conditions imposed by some measurements - because the 4-manifold depends upon these boundary conditions too! If an initial state is assumed then it must be described in a way that depends upon one type of measurement *or* another type, yet the state description is able to predict the result of any measurement even if the state is described in terms of an incompatible measurement. This peculiar way of describing the result of a state preparation without



knowledge of the subsequent measurement is *quantum mechanics!*

The asymptotic form of the metric is *not* determined by the state preparation and will be affected by future experiments. This contradicts the usual interpretation of general relativity, that the metric is well defined at any space-time point on the manifold. It is, however, an inescapable conclusion in almost all interpretations of quantum mechanics.<sup>2</sup>

## Comparison

Both classical and 4-geons are solutions of Einstein's equations of general relativity subject to boundary conditions imposed by experimental apparatus. In the first case there is a well-defined time coordinate: initial state and evolution have their usual meaning; evolution is deterministic and cannot depend upon future events (nor space-like separated events). In the latter case, time has no global meaning and the concepts of state preparation, evolution and time are dubious or at best asymptotic approximations; measurements made at a future time (as defined asymptotically) are valid as boundary conditions as much as the state preparation itself.

## 6.7 Non-classical Behaviour as a Boundary Value Problem

A simple, well understood classical example can help to interpret what is happening in a 4-geon.

Consider a machine that throws identical balls from one side of the room to another. The balls follow a certain trajectory which is determined by the initial conditions - position and velocity of the throw. If the initial conditions are not precisely fixed then a stochastic element will be introduced, and the trajectory cannot be predicted because we lack some information - in other words some of the variables are hidden. If a bucket is placed on the far side of the room some balls may land in it and some would miss it. The example deliberately uses a machine

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<sup>2</sup>Bohm's theory may be an exception.

to avoid any suggestion that the throw is adjusted in order to hit the bucket. If the throw is precisely defined then the balls will either land in the bucket or miss it. If there is some variability, then some may land in the bucket while others will miss it. What is certain is that the trajectory of the ball does *not* depend upon the position of the bucket; it is well described if the initial position and momentum are known. Stating where it hits the far side of the room is a superfluous boundary condition which is either redundant or contradictory.

To make it even clearer that the position where the ball lands on the far side of the room does not depend upon the position of the bucket, the bucket could be moved after the throw but before the ball lands; clearly the trajectory would be unaltered. Even if the initial position and velocity are unknown, they still have well-defined values which determine the trajectory uniquely.

A contrasting classical experiment uses a string stretched from one side of the room to the other (initially straight and still), held at one end by a shaking machine (or person) What is the movement (shape at different times) of the string? It certainly depends upon the shaking machine but now it *can* depend upon what is happening at the other end too - indeed the shape cannot be calculated without knowing *more* than just what happens at the end with the shaking machine. If a function  $a(t)$  describes the vertical displacement of one end and  $b(t)$  the displacement at the other end then the shape of the string is well defined. Some combinations of  $a(t)$  and  $b(t)$  may be incompatible (*ie.* break the string), but in general these two functions are required to define the movement of the string. This is in marked contrast to the previous example. In particular, consider a point half-way across the room; the position of the balls here is determined only by the throwing mechanism on the near side of the room, whilst the position of the string depends upon what is happening at *both* ends of the room.

Although the difference in the two systems can be analysed in terms of the differential equations which govern the system, a simple qualitative explanation is



sufficient to see the fundamental difference between the two cases. In the second case only, there is a route for information to pass from the far side of the room back towards the middle and even to the near side. The information passes, as a wave, along the string and the shape of the string can be regarded as the superposition of waves travelling in the two directions. In the case of the balls there is no mechanism to pass information from the far side of the room to the middle: no way in which the position of the bucket could influence the trajectory. Of course, if a person rather than a machine had been used to throw the balls the simple mechanistic picture would fail because the thrower could see where the bucket was and use that information to choose the position and velocity of the throw; - even in this case, however, there is no mechanism for the trajectory to change once the ball has been thrown.

The 4-geon can be regarded as having the property of the string in the sense that there is a causal link between the *future* measurement conditions and the *initial* state preparation boundary conditions. Information can, as it were, travel back down the structure that is the particle, allowing conditions which an outside observer sees as past and future to define the manifold. Thus we see that not only do 4-manifolds require extra boundary conditions to fully define what is happening but that the breakdown of the causal structure enables *future* boundary conditions to influence the evolution.

To prevent a total breakdown of causality in nature it is appealing to regard the particle as a tube or string along which, or within which, information can travel without regard to the normal conventions of past and future.

This picture in terms of boundary conditions explains one experimentally known fact about quantum mechanics; variables are complementary if, and only if, it is impossible to simultaneously measure them. This may seem obvious but it is not necessarily so. It is a fact that quantum mechanics has both quantum observables, like position and momentum, and properties which have a classical behaviour such as



lepton number; these are normally dealt with by using a Hilbert space formalism and then applying superselection rules. There is no rule in quantum mechanics which predicts which observables are classical and which obey non-classical logic. The fact that two observables cannot be experimentally measured at the same time is fully compatible with them being classical properties which have definite but unknown values - this would be a simple hidden variables theory. Even if two observables cannot be simultaneously measured in theory, they could still be purely classical properties of the object.

In the early days of quantum physics it was believed by many scientists, Einstein among them, that quantum physics reflected our ignorance of a particles properties rather than being a full description of nature, but quantum mechanics as experimentally verified is not consistent with this view.

It is certainly a necessary condition that measurements cannot be made simultaneously for observables to be incompatible, otherwise the experiment could be set up and values obtained simultaneously. However, the impossibility of making simultaneous measurements is not sufficient for quantum mechanical incompatibility to be a logical necessity. The 4-geon description does require that where boundary conditions associated with two measurements are incompatible then the observables cannot be compatible. The only proviso is that the boundary conditions are not redundant; this is a consequence of the *initial* conditions not being sufficient and the assumption that the measurements do impose further, useful, boundary conditions.

## Chapter 7

# Construction of an Orthomodular Lattice in General Relativity

### 7.1 Construction of a Modular Lattice

By considering the measurements of the  $x$  and  $y$  components of spin of a 4-geon with spin-half it is possible to construct a modular lattice of propositions. In this chapter we will reconsider each of the properties of a modular lattice in turn (as described in section 2.8) and show that they are satisfied by this particular example. For the construction which follows, we require the 4-geon to have more than one possible outcome from a Stern-Gerlach apparatus. We will consider two possible outcomes ( $> 0, \leq 0$ ); the exact spectrum, whether it is finite or infinite, continuous or discrete, is not important. The choice of  $x$  and  $y$ -spin, and the restriction to two outcomes, is made to give a simple model of the spin for a spin-half particle; momentum and position could equally well have been used.

The relationship between orthomodular lattices and complex Hilbert spaces described in references[7, 21], means that once we have constructed an orthomodular

lattice of propositions we can apply the internal symmetries and the symmetries of spacetime in the usual way (see for example [5, chapter 3]) to determine the form of the operators and the eigenvalues for spin, momentum *etc.* The fact that a spin-half particle has two possible values for the  $x$ ,  $y$  or  $z$  component of spin need not be assumed.

The set of all possible 4-geon manifolds,  $\tilde{\mathcal{M}}$ , is not very useful, since it includes manifolds compatible and incompatible with every experimental arrangement. Let us constrain the possible manifolds by setting up the state preparation apparatus as depicted in figure 7.1. By Axiom 3 (see section 6.3), the apparatus imposes boundary conditions which limit the set of relevant manifolds to  $\mathcal{M} \subset \tilde{\mathcal{M}}$ , *ie.* to those 4-geons compatible with the apparatus of figure 7.1.

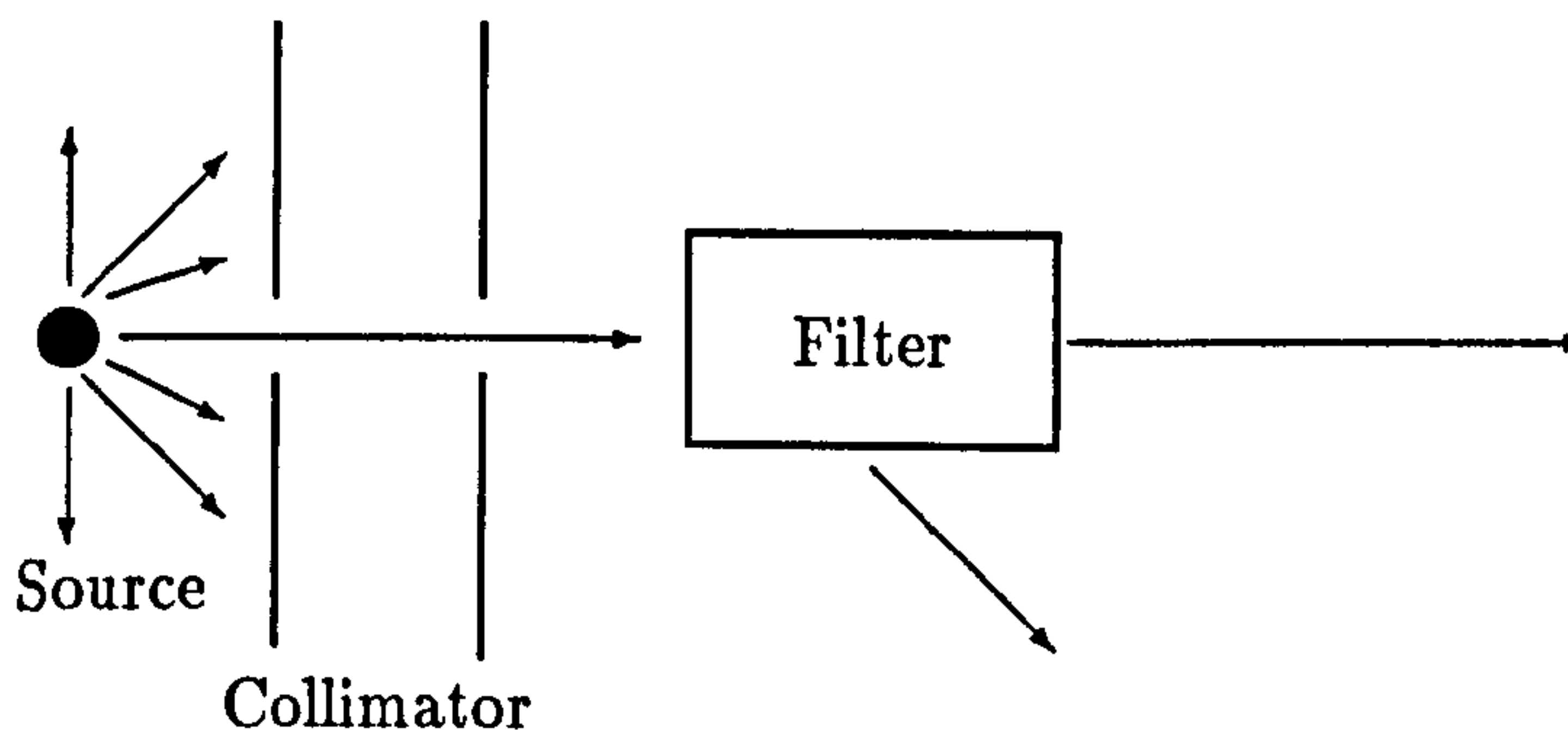


Figure 7.1: The boundary conditions imposed by state-preparation

Next we can set up a Stern-Gerlach apparatus aligned with the  $x$ -axis, followed by an  $x$ -position detector which here gives a value for the spin (see figure 7.2). By Axiom 2 (section 6.2), the particle will certainly be detected at one position and only one position. We denote by  $\mathcal{X}$  the 4-geon manifolds consistent with the state-preparation and the  $x$ -oriented Stern-Gerlach equipment. Clearly  $\mathcal{X} \subseteq \mathcal{M}$ ; because of the 4-geon postulate we can have  $\mathcal{X} \neq \mathcal{M}$  (as shown in figure 7.4). The fact that  $\mathcal{X} \neq \mathcal{M}$  is, by itself, non-classical; the measurement does not partition the results



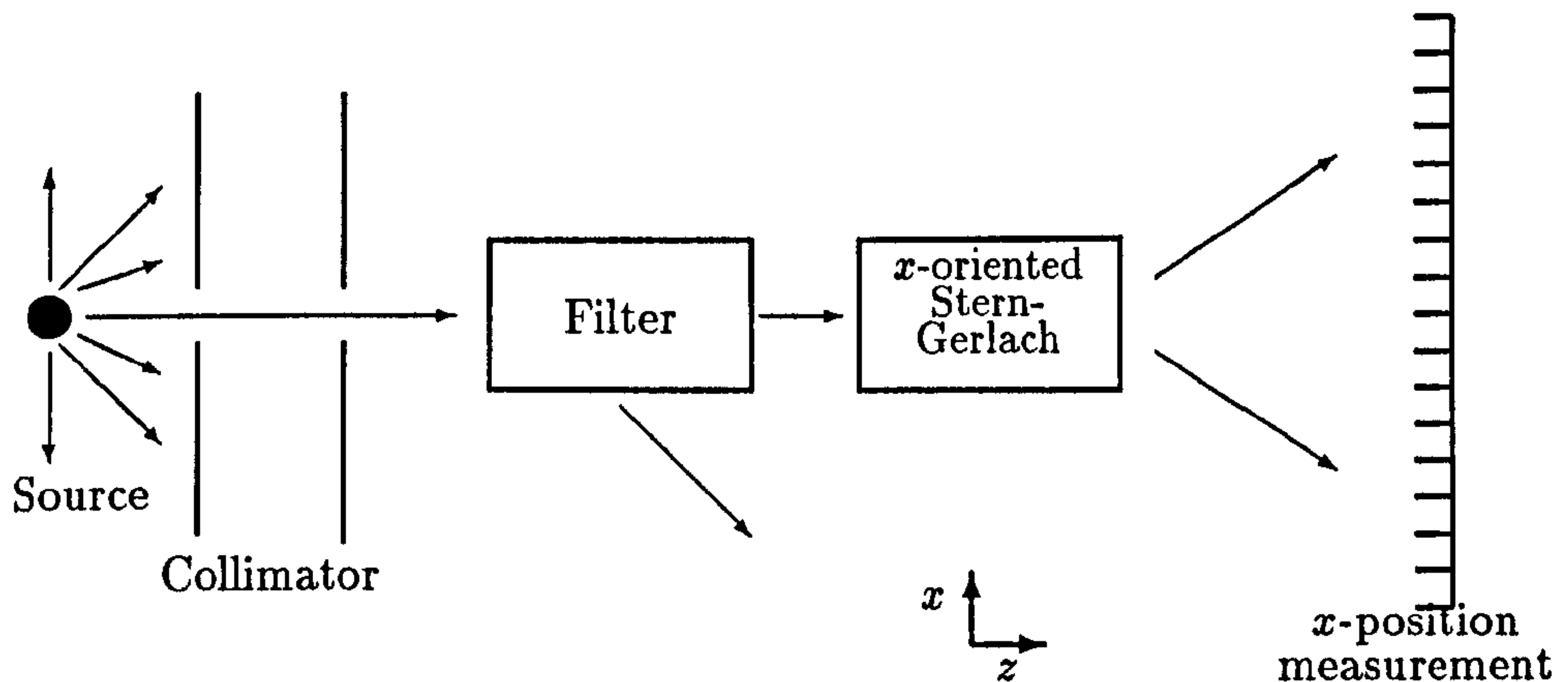


Figure 7.2: The boundary conditions imposed by state-preparation and an  $x$ -spin measurement

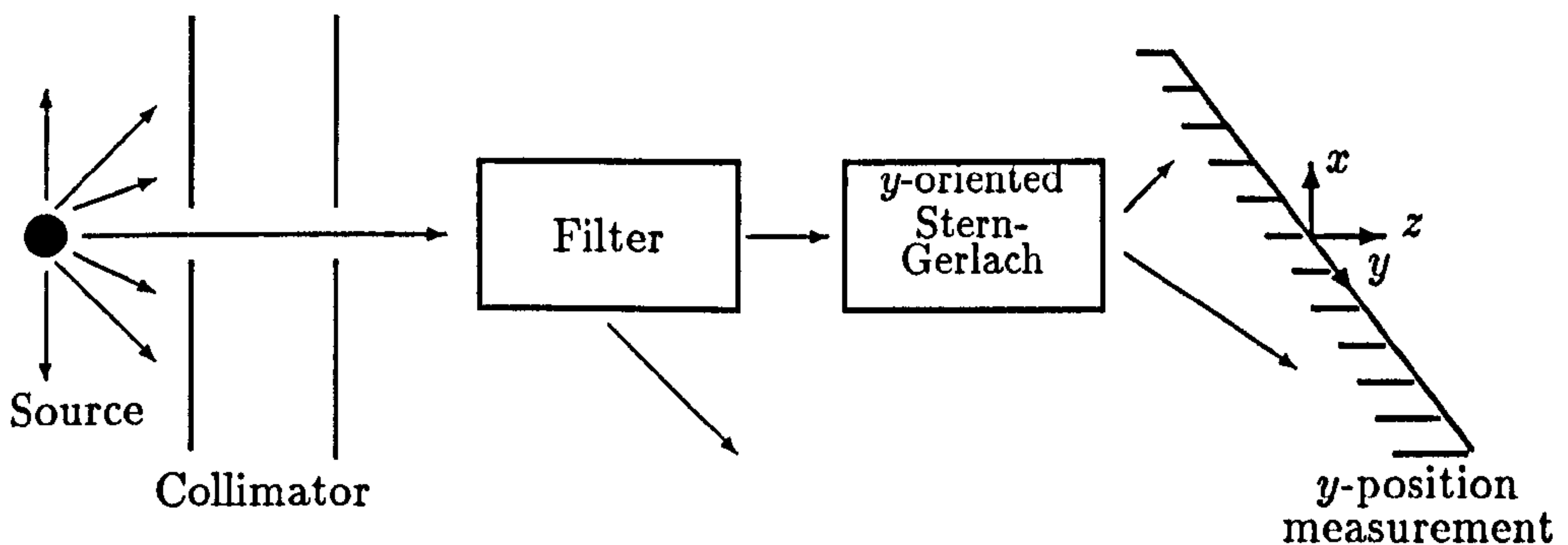


Figure 7.3: The boundary conditions imposed by state-preparation and a  $y$ -spin measurement

of the state preparation, but adds further constraints. Of all the manifolds in  $\mathcal{X}$ , some will correspond to  $x > 0$ , and the remainder to  $x \leq 0$ ; these will be denoted  $\mathcal{X}^+$  and  $\mathcal{X}^-$ , respectively. Note that the same measurement apparatus determines  $x > 0$  and  $x \leq 0$ : therefore  $\mathcal{X} = \mathcal{X}^+ \cup \mathcal{X}^-$ .

A  $y$ -axis measurement may be made in a similar way (see figure 7.3) which defines subsets  $\mathcal{Y}$ ,  $\mathcal{Y}^+$  and  $\mathcal{Y}^-$  of  $\mathcal{M}$ . An  $x$  and  $y$ -oriented Stern-Gerlach apparatus clearly cannot both be set in the *same* place at the *same* time; they are incompatible, and by Axiom 4 the boundary conditions which they set are incompatible. Hence

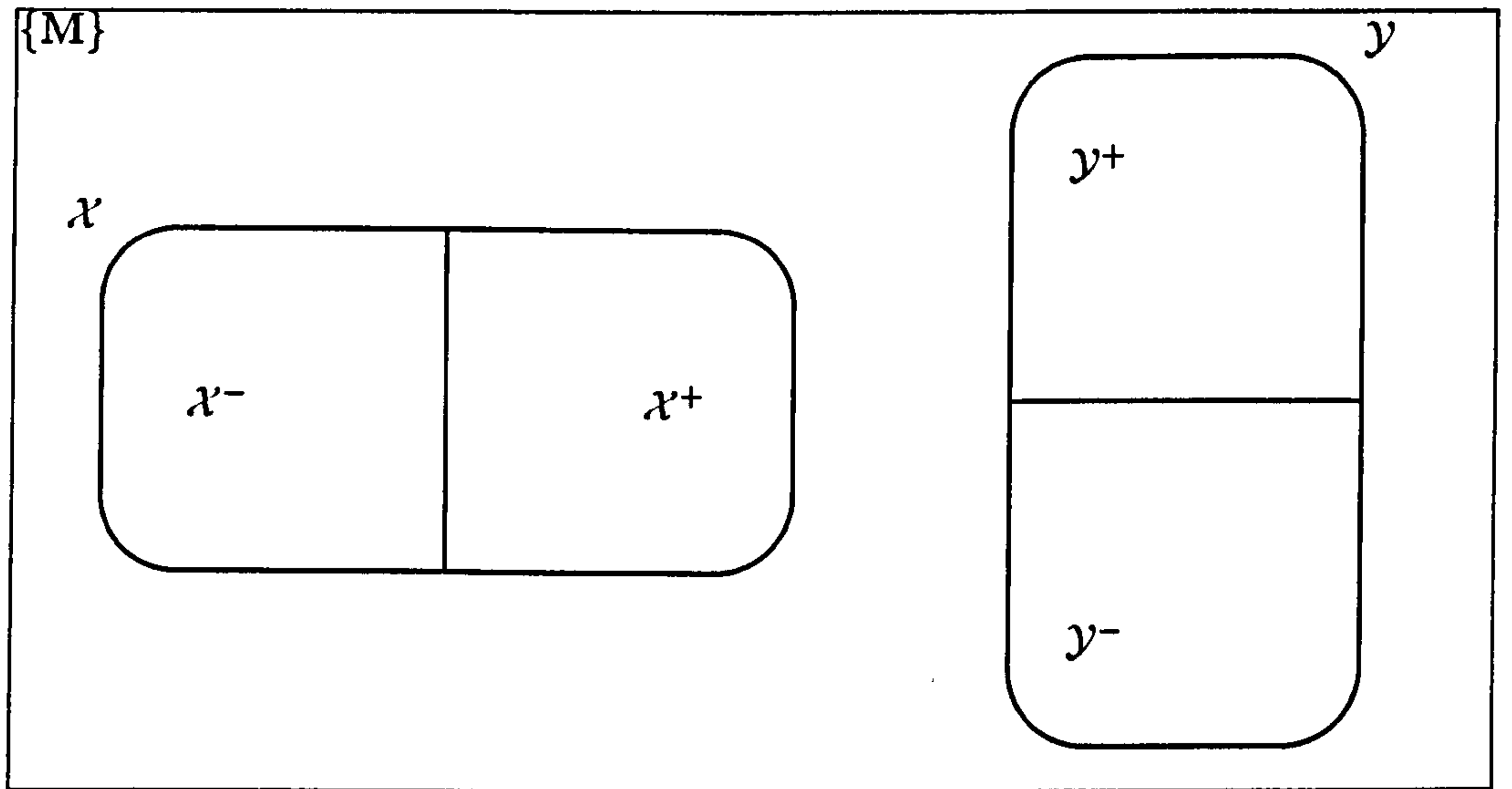


Figure 7.4: Sets of 4-manifolds corresponding to measurements of  $x$  and  $y$  components of spin.

$\mathcal{Y}$  and  $\mathcal{X}$  are disjoint subsets of  $\mathcal{M}$ .

### The Propositions

The propositions are the equivalence classes of outcomes of yes/no experiments, two experiments being equivalent if there is no state preparation that can distinguish them. Four non-trivial propositions,  $p, q, r$  and  $s$ , can be stated about the  $x$  and  $y$ -spin of 4-geon manifolds,  $M$ . They listed in table 7.1, together with the subsets of manifolds in the equivalence class and the experimental results which they relate to.

In addition, there are the two trivial propositions  $0$  and  $I$ . Axiom 2 ensures that there exists at least one 4-geon manifold consistent with any measurement ( $\exists M \in \mathcal{X}$ ). Equivalently, given the state-preparation and measurement boundary conditions then  $M \in \mathcal{X}$ . The trivial propositions,  $I$  (which is always true) and  $0$

(which is always false), correspond to this Axiom and its converse:

$$\begin{aligned}
 0 &\equiv M \in \emptyset & I &\equiv M \in \mathcal{X} \text{ for an } x\text{-spin measurement} \\
 &\equiv \mathcal{X} = \emptyset & &\equiv M \in \mathcal{Y} \text{ for a } y\text{-spin measurement} \\
 &\equiv \mathcal{Y} = \emptyset & &
 \end{aligned} \tag{7.1}$$

The fact that the trivial propositions have more than one interpretation is also common to classical mechanics. For example, the propositions that *the momentum is a real number* and that *the position is a real number* are both always true for a classical object. What is non-classical here is that these two physical descriptions correspond to two different (and disjoint) sets of possible results - two disjoint sets of manifolds. Classically, the measurements are different ways of partitioning the common set defined by the initial conditions alone. Here the measurements define two different sets, but the propositions are identical because the sets give the same information.

Proposition	Manifolds	Measurement
0	$\emptyset$	Always False
$p$	$M \in \mathcal{X}^+$	The $x$ -Spin is measured to be $> 0$
$q$	$M \in \mathcal{X}^-$	The $x$ -Spin is measured to be $\leq 0$
$r$	$M \in \mathcal{Y}^+$	The $y$ -Spin is measured to be $> 0$
$s$	$M \in \mathcal{Y}^-$	The $y$ -Spin is measured to be $\leq 0$
$I$	$M \in \mathcal{X}$	The $x$ -Spin is measurable
$I$	$M \in \mathcal{Y}$	The $y$ -Spin is measurable

Table 7.1: The propositions and sets of manifolds of the spin-half system

### Partial Ordering

The ordering relation for two propositions,  $a$  and  $b$ , is  $a \leq b$  which means that  $a$  true implies that  $b$  is true. For the spin-half system the partial ordering is almost trivial:

$$0 \leq p \leq I, \quad 0 \leq q \leq I, \quad 0 \leq r \leq I, \quad 0 \leq s \leq I \tag{7.2}$$



In this case there can be no ordering between  $p$  and  $r$  *etc.* when they are in different directions, because  $\mathcal{X}$  and  $\mathcal{Y}$  are disjoint (and can clearly be distinguished by some state preparations) and so a manifold cannot be in both. The propositions of the system therefore form a poset (partially ordered set). Generally, the ordering relation can only be applied to propositions if there exists at least one experimental arrangement which evaluates both of them together.

### Meet and Join

The meet of two propositions,  $a \wedge b$ , is the largest proposition, the truth of which implies that both  $a$  and  $b$  are true. For any poset it follows that:  $a \wedge a = a$ ,  $a \wedge I = a$  and  $a \wedge 0 = 0$ . For this system we have in addition:

$$a \wedge b = 0, \quad \forall a \neq b \quad (7.3)$$

For a 4-geon manifold,  $M$ , to be in the meet of  $p$  and  $r$  ( $p \neq r$ ), it would have to be in  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  which is not possible; the solution set is, therefore, the empty set which corresponds to 0. Membership of the subsets  $\mathcal{X}^+$  and  $\mathcal{Y}^+$  corresponds to physically distinguishable statements about the state preparation so they are distinct propositions (the equivalence relation does not affect this conclusion).

The join of two propositions,  $a \vee b$ , is the smallest proposition which is true whenever either  $a$  or  $b$  is true. For any poset it follows that:  $a \vee a = a$ ,  $a \vee 0 = a$  and  $a \vee I = I$ . For this system we have in addition:

$$a \vee b = I \quad \forall a \neq b \quad (7.4)$$

In this small system there is no other acceptable choice for  $p \vee r$  *etc.*

### Orthocomplementation

As in classical mechanics we consider the orthocomplementation  $a^\perp$  of a proposition  $a$  by taking the set-theoretic complement with respect to all possible outcomes of the same experiment. We define the complements of our system in table 7.2

Complement	Manifolds
$0^\perp = I$	Always True
$p^\perp = q$	$M \in (\mathcal{X} \setminus \mathcal{X}^+ \equiv \mathcal{X}^-)$
$q^\perp = p$	$M \in (\mathcal{X} \setminus \mathcal{X}^- \equiv \mathcal{X}^+)$
$r^\perp = s$	$M \in (\mathcal{Y} \setminus \mathcal{Y}^+ \equiv \mathcal{Y}^-)$
$s^\perp = r$	$M \in (\mathcal{Y} \setminus \mathcal{Y}^- \equiv \mathcal{Y}^+)$
$I^\perp = 0$	Always False

Table 7.2: The complements of the propositions of the spin-half system

From table 7.2 and table 7.1, it is clear that the required properties of orthocomplementation are satisfied:

1.  $(a^\perp)^\perp = a$
2.  $a \vee a^\perp = I$  and  $a \wedge a^\perp = 0$
3.  $a \leq b \Rightarrow b^\perp \leq a^\perp$

The first two follow directly from set theory, while the third only applies in the cases:  $a < I$  or  $0 < a$ , because of the simple structure of this poset.

The definition given satisfies DeMorgan's laws:

$$(a_1 \wedge a_2)^\perp = a_1^\perp \vee a_2^\perp \quad (7.5)$$

$$(a_1 \vee a_2)^\perp = a_1^\perp \wedge a_2^\perp \quad (7.6)$$

$$(7.7)$$

Thus we have an orthocomplemented poset. DeMorgan's Laws can be used to define the join of two incompatible propositions in terms of the meet and orthocomplementation *eg.*:

$$p \vee r = (q \wedge s)^\perp = 0^\perp = I \quad (7.8)$$

## Lattice

A lattice is a poset where the meet and join always exist. The meet and join of any two elements of this system always exist, these being 0 and I respectively, for any two different propositions. Table 7.3 shows the meet and join for all the propositions.

$\wedge$	0	$p$	$q$	$r$	$s$	$I$
0	0	0	0	0	0	0
$p$	0	$p$	0	0	0	0
$q$	0	0	$q$	0	0	0
$r$	0	0	0	$r$	0	0
$s$	0	0	0	0	$s$	0
$I$	0	0	0	0	0	$I$

$\vee$	0	$p$	$q$	$r$	$s$	$I$
0	0	$I$	$I$	$I$	$I$	$I$
$p$	$I$	$p$	$I$	$I$	$I$	$I$
$q$	$I$	$I$	$q$	$I$	$I$	$I$
$r$	$I$	$I$	$I$	$r$	$I$	$I$
$s$	$I$	$I$	$I$	$I$	$s$	$I$
$I$	$I$	$I$	$I$	$I$	$I$	$I$

Table 7.3: The meets and joins of the propositions of the spin-half system

The poset is thus seen to be an orthocomplemented *Lattice*.

## Orthomodularity

The orthomodularity condition:

$$a \leq b \Rightarrow b = a \vee (b \wedge a^\perp) \quad (7.9)$$

is satisfied by the simple spin-half poset, as can be seen by considering each case,  $\forall a \in \{p, q, r, s\}$ :

$$0 \leq a \Rightarrow a = 0 \vee (a \wedge I) \quad (7.10)$$

$$a \leq I \Rightarrow I = a \vee (I \wedge a^\perp) \quad (7.11)$$

$$a \leq a \Rightarrow a = a \vee (a \wedge a^\perp) \quad (7.12)$$

$$0 \leq I \Rightarrow I = 0 \vee (I \wedge I) \quad (7.13)$$



## Modularity

That this lattice is modular can be seen by examining it case by case. The failure of the modularity law, as required for a strictly orthomodular lattice, will only occur for systems with an infinite spectra [24, page 220].

## Distributivity

A simple counterexample suffices to show that the distributive rule fails for propositions about different directions:

$$p \wedge (r \vee r^\perp) \neq (p \wedge r) \vee (p \wedge r^\perp) \quad (7.14)$$

the LHS is  $p \wedge I = p$ , while the RHS is  $0 \vee 0 = 0$ ; thus  $p$  and  $r$  are not compatible. The result can be checked from table 7.3 of meets and joins or by noting that the subsets  $\mathcal{X}^+, \mathcal{Y}^+, \mathcal{Y}^-$  corresponding to the propositions  $p, r$  and  $s$ , respectively, are all disjoint and not related by the equivalence relation.

## Atomicity

An atom is a proposition, different from 0, which does not have any smaller proposition. The propositions  $p, q, r, s$  are clearly the atoms.

## Covering Property

We say that  $a$  covers  $b$  if  $a > b$ , and  $a \geq c \geq b$  implies either  $c = a$  or  $c = b$ . A lattice has the covering property if the join of any element,  $a$ , with an atom,  $t$ , not contained in  $a$  covers  $a$ . Clearly  $\forall a, b \in \{p, q, r, s\}$ :

$$0 \vee a = a \quad \text{which covers } 0 \quad (7.15)$$

$$a \vee b = I \quad \text{which covers } a \quad (7.16)$$

This establishes that the system has the covering property.

Starting with propositions about sets of manifolds in classical general relativity, we have constructed a non-distributive, orthomodular lattice, which is atomic and has the covering property (this example is also modular). The significance is not just that such a lattice is a feature of quantum mechanics, but that it is *the* distinguishing feature of quantum mechanics. It has previously been thought that a non-distributive lattice of propositions could never be constructed from a classical theory and hence that no classical explanation of quantum mechanics was possible; this is shown to be false. The present work thus gives a classical explanation for the origin of quantum mechanics and because it is based on possibilities offered by the accepted theory of general relativity, it offers the most economical interpretation.

## 7.2 A Lattice is Always Defined

An argument due to Mackey ([28] or see [7, chapter 13] for a clear summary) shows that the accepted idea of a measurement leads naturally to the requirement for an orthomodular poset. His paradigm for a measurement is equivalent to the description in section 7.1. He assigns measured parameters to the position of a needle on a measuring apparatus, while I use the position in a position detector. Since the measurement apparatus sets boundary conditions (which is an essential part of the argument in this thesis), then orthomodularity for the general case follows by the same argument as Mackey. However, his argument is limited to defining meet and join of compatible propositions and says nothing about the meet and join of incompatible ones - hence he can say only that the logical structure is a poset - not a lattice.

This work goes further. The meet of incompatible propositions is defined by the intersection of sets of manifolds, it is generally the zero element. Incompatible measurements set incompatible boundary conditions and define corresponding disjoint sets of possible manifolds. There can be no manifolds common to both. Using De Morgan's laws the join of two incompatible propositions can be defined in terms

of the meet:

$$a \vee b = (a^\perp \wedge b^\perp)^\perp \quad (7.17)$$

The only problem with this definition of meet, and hence join, is for atomic propositions of continuous spectra. They are mathematical idealisations, and so we cannot appeal to experimental arrangements for a definitive answer. An important consequence of this limitation is that we cannot prove that the covering property applies. A circular argument only shows that meets and joins with incompatible *atomic* propositions can be consistently defined:

Step 1 The measurable propositions form a modular lattice.

Step 2 The lattice is represented by projections of a Hilbert space.

Step 3 A mathematical idealisation is made.

Step 4 Operators for incompatible observables are derived.

Step 5 Operators for atomic propositions are known.

Step 6 Meet and join of atomic propositions and other mathematical idealisations are calculated.

Step 7 The covering property is seen to be satisfied.

Step 8 The commutators of operators can be examined.

Step 9 The compatibility of measurements can be inferred from the compatibility of operators.

Unfortunately, the mathematical justifications for the move from step 1 to step 2, which attempt to show that it is a unique representation, require a complete lattice with the covering property which is only deduced in step 7. So the results are consistent but claims of uniqueness are undermined.



There remains a possibility that careful examination of experimental arrangements will lead to an unambiguous definition of the meet and join of mathematically idealised propositions. That may not be as easy as it sounds, as can be seen by the consideration of position and momentum measurements:

- Simple physical arguments show that there is a limit to simultaneous position and momentum measurements - the well known arguments of Bohr[36] can be cited, for example. In all conceivable experiments  $\Delta p \Delta x \approx \hbar$  seems to be the smallest theoretically obtainable limit.
- The states are defined by elements,  $\Psi(x)$ , of a Hilbert space
- Real numbers are used to represent position and momentum.
- Momentum is identified with  $\partial_x$  and position with  $x$ .
- It is seen that the propositions  $P \in [\alpha, \beta]$ , which we call  $a$ , and  $x \in [\gamma, \delta]$ , which we denote  $b$ , are incompatible for all finite intervals  $[\alpha, \beta]$  and  $[\gamma, \delta]$ . So the theory now predicts that it is impossible to ascertain simultaneously that *the object is in the room* and that *the object has a speed less than 100m/s!* Which goes far beyond the original experimental arguments.

The covering property is satisfied by a Hilbert space [7, page 107], which means that the join of a finite proposition about momentum and an atom of position measurement cannot be  $I$  as is the case for finite intervals *eg.*: Let  $c$  be the atomic proposition that *the object has an x-position of 2* and  $a$  and  $b$  defined as above, then although  $a \vee b = I$ ,  $a \vee c \neq I$ . Using De Morgan's law gives  $a^\perp \wedge c^\perp \neq 0$ . In other words  $a^\perp$  and  $c^\perp$  commute[10]. This suggests that it could be physically possible to determine simultaneously that the momentum is in a well defined range and that the object is not at a certain position (*eg.*  $x = 2$  in this example).

Although the limits of experimental arrangements provided the initial motivation for using a Hilbert space structure, the mathematical structure of such a space

then provided further, unexpected, information about the limits of measurements, as depicted in figure 7.5.

### 7.3 Extensions to Observables with a Continuous Spectrum

The previous section constructed a modular lattice of propositions for measurements of a 4-geon. The orthomodularity condition followed naturally from the requirement that the ordering relation applied only to propositions which could be determined together. By inspection, the modularity condition was also satisfied for this example. An argument due to Mackey[28] shows that for incompatible observables with unbounded continuous spectra the lattice cannot be modular. The argument relies upon the property of  $\cap$ -continuity  $\cup$ -continuity and a theorem due to Kaplansky[25].

**Definition 7.1 ( $\cup$ -continuity)** *A Lattice is  $\cup$ -continuous if for every non-increasing sequence  $\dots a_{n-1} \geq a_n \geq a_{n+1} \geq \dots$  and every  $b$ :*

$$b \cup \left( \bigcap_n a_n \right) = \bigcap_n (b \cup a_n) \quad (7.18)$$

**Definition 7.2 ( $\cap$ -continuity)** *A Lattice is  $\cap$ -continuous if for every non-increasing sequence  $\dots a_{n-1} \leq a_n \leq a_{n+1} \leq \dots$  and every  $b$ :*

$$b \cap \left( \bigcup_n a_n \right) = \bigcup_n (b \cap a_n) \quad (7.19)$$

**Theorem 7.3 (Kaplansky)** *A complete orthocomplemented, modular lattice is  $\cap$ -continuous and  $\cup$ -continuous.*

See [25] for a proof.

We will show that the lattice for propositions about position and momentum cannot be modular and must therefore be orthomodular. The proof, and the result, is readily applicable to any two incompatible observables with infinite spectra.

Following Jauch[24] we construct an increasing sequence of propositions:  $a_n = \text{the } x\text{-position is in the range } [-n, n]$ . Clearly  $\cup_n a_n = I$ . Let the proposition  $b$  be *the momentum is in the range  $[\alpha, \beta]$* . States do not exist in which the position is known to be in a finite closed interval and also the momentum has a known finite range, therefore;

$$b \cap a_n = 0 \quad \forall n \quad (7.20)$$

It follows that:

$$b \cap \left( \bigcup_n a_n \right) = b \cap I = b \quad (7.21)$$

while:

$$\bigcup_n (b \cap a_n) = \bigcup_n (0) = 0, \quad (7.22)$$

hence the Lattice is not  $\cap$ -continuous, and by Kaplansky's theorem it cannot be modular.

Jauch proves equation 7.20 by using the Hilbert space formalism and the properties of Fourier transformations, which relate position and momentum representations. To apply the same method to a 4-geon without assuming a Hilbert space representation we would need to prove that the experimental arrangement to evaluate  $a_n$  is incompatible with those to evaluate  $b$ . Furthermore, it would be necessary to show that at least one state preparation existed which could distinguish  $a_n$  and  $b$  to show that they were indeed distinct as propositions as well as being different manifolds. This goes considerably beyond the thought experiments of Bohr [36] which show that  $\delta p \delta q \geq \hbar$  (see figure 7.5 for an illustration of the interplay between theory and experiment). It is not always appreciated that quantum mechanics forbids us knowing simultaneously that: the electron is definitely in the room (*ie.*  $0 < x < 5m$ ) and knowing that the momentum is in a finite range (*eg.*  $p_x < 10kgm/s$ ) even though the product of the uncertainties is  $50kgm^2/s$  - rather more than Planck's constant!



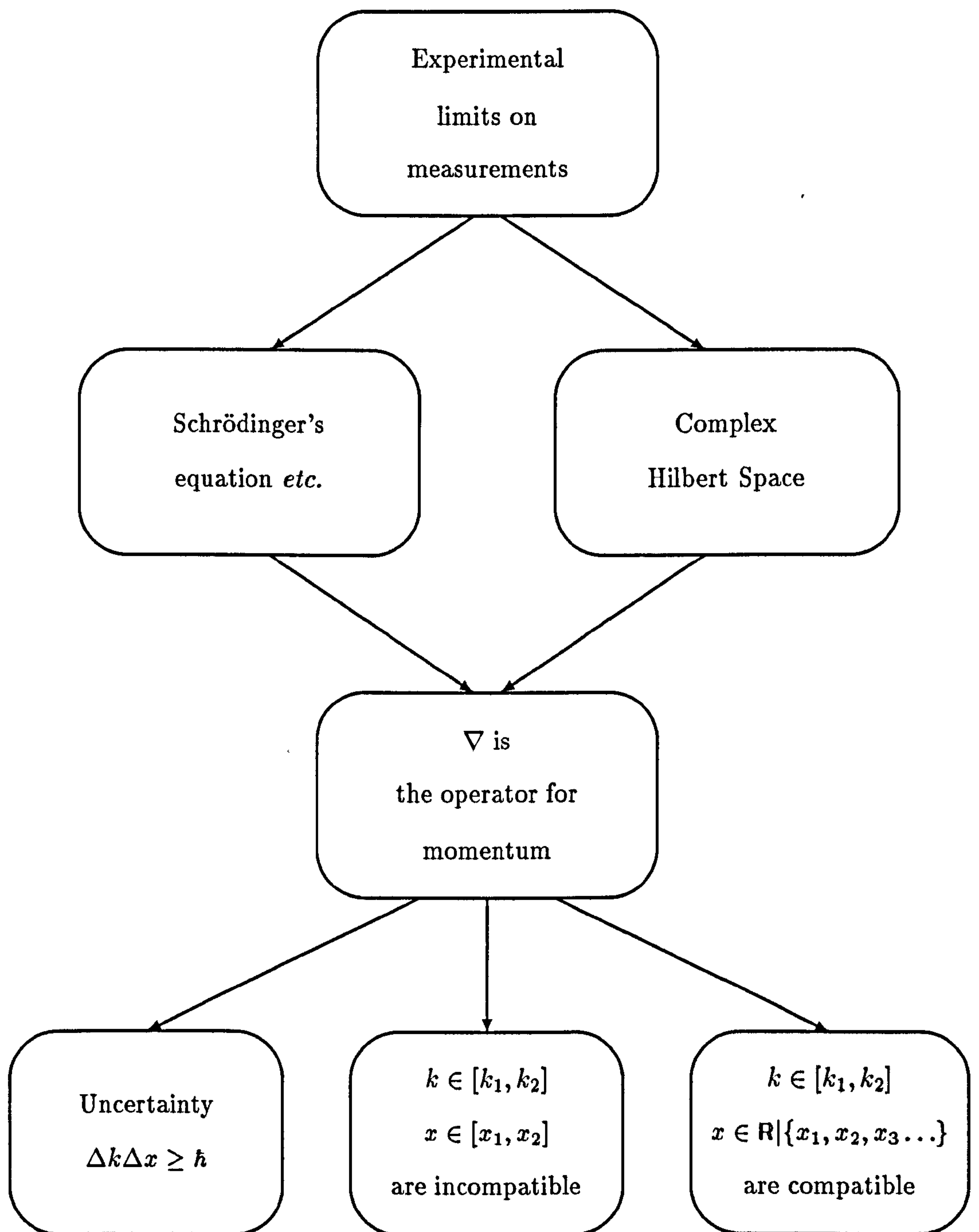


Figure 7.5: The experimental limits on measurements are formulated in quantum mechanics; which gives, not only the required uncertainty relations, but also a much stronger incompatibility (bottom centre) with theoretically compatible observables (bottom right).

## Chapter 8

# Spin-half and General Relativity

Many elementary particles (the fermions) have half-integral spin and must be described by a mathematical object with the appropriate transformation properties *ie.* a spinor. The fact that a spinor (rather than a scalar or vector) is needed to describe a fermion leads to the Dirac equation (rather than the Klein-Gordon or massive vector field equations). The spin of a particle determines how a description of it must transform under rotations: with half-integral spins for particles which need to be described in a different way after a rotation of  $2\pi$ . Gravitational waves are spin-two, as can be seen by examining the transformation properties of a plane wave (see for example [39]). To model a fermion using the theory of general relativity thus requires the construction, from an intrinsically spin-two theory, of an object which transforms as a spinor.

An article by Friedman and Sorkin[15] showed how a manifold with the characteristics of a spinor could be constructed. This was a significant development, because general relativity was regarded as, exclusively, a spin-two field theory before their paper was published. However, the freedom to choose the topology of the manifold is enormous - there are manifolds with homotopy classes for every conceivable group. Their paper gives a general argument for how half-integral spin can arise, together with one specific example of a manifold with the properties of a

spinor under rotations of the ‘inner’ region.

## 8.1 Friedman and Sorkin’s Paper

The following is a synopsis of the essential features of their paper which concentrates on the classical aspects (they saw the result as being applicable to quantum gravity only).

General relativity is a theory constructed from elements of the tangent space to a manifold. Vectors defined on a manifold are elements of the tangent space at a point  $m$  of the manifold,  $T_m M$ , while one-forms are elements of the dual space  $T_m^* M$ . The metric tensor is an element of  $T_m^* M \otimes T_m^* M$ . Consequently, they all transform as representations of  $SO(3)$  under rotations of the tangent space. The tangent space itself can be constructed from a coordinate chart: if  $\psi$  is a chart from a manifold,  $M$ , to  $R^n$  then  $d\psi^{-1}$  is a map between tangent spaces, *ie.* from  $R^n$  to  $T_m M$ . All such constructions will be invariant under a rotation by  $2\pi$

The programme to describe elementary particles as topological structures in space-time using general relativity is criticised because only integral spin entities can be constructed this way [31]. A paper by Friedman and Sorkin [15] shows how spin-half could be achieved in quantum gravity.

Their paper uses the following concepts:

1. Asymptotically flat 3-manifolds,  $M$ .
2. Asymptotically trivial diffeomorphisms,  $D$  - a diffeomorphism from  $M$  to  $M$  which reduces to the identity at infinity. This includes all diffeomorphisms of compact support.
3. Equivalence classes of metrics  $[g]$ , with the equivalence relation being the asymptotically flat diffeomorphisms,  $D$ .
4. Equivalence classes of asymptotically trivial diffeomorphisms,  $[D]$ , with the



equivalence relation being homotopy equivalence relative to infinity (ie. a set  $[D] \times I$ ). The element containing the identity is  $[D_0]$ .

5. Rotations of the manifold are defined as diffeomorphisms which are asymptotically rotations. At infinity the manifold becomes Euclidean and in this region rotations are well-defined and consistent with the classical meaning.

6. A state vector,  $\psi$ , which is a function of the metric  $\psi : g \rightarrow \mathbb{C}$  and satisfies:

(a)  $\forall \chi$  in  $[D_0]$  then  $\psi(\chi g) = \psi(g)$

(b) If  $[g]$  and  $[g']$  are two different classes, then there is some  $\psi$  such that  $\psi(g) \neq \psi(g')$ .

(c)  $\psi$  transforms under rotations as:

$$\psi(g) \rightarrow L_\alpha(\theta)\psi(g) = \psi(R_\alpha(\theta)g), \quad (8.1)$$

a representation which may be double valued.

The spinor nature of  $\psi$  arises when  $L_\alpha(\theta)$  (abbreviated  $L(\theta)$ ) cannot be extended to the entire manifold. For then  $[L(2\pi)g] \neq [g]$  and by item 6b above there exists  $\psi' = L(2\pi)\psi \neq \psi$ .

Manifolds in which  $R(2\pi)$  is in  $D_0$  have been characterised by Hendriks [19]. A counter-example can be constructed by removing from  $\mathbb{R}^3$  a solid cube and identifying opposite faces of its boundary after a  $90^\circ$  rotation.

Friedman and Sorkin's construction is closely related to the Misner, Thorne and Wheeler description of a spinor using a cube-in-a-room (see [30][page 1148]). If the frame is rotated by  $2\pi$  while keeping the interior cube fixed then we have an element of  $D$  - but is it in  $[D_0]$ ? In the physically familiar case it is. Simply release the inner cube and allow it to 'unwind' (rotate in the opposite direction) until the system is back to the identity. Clearly this is a homotopy parameterised by the angle of rotation of the inner cube where each diffeomorphism is asymptotically trivial in that it leaves the frame stationary. The interesting manifolds are where the rotation

cannot be extended from the frame through to the whole structure *ie.* the inner cube cannot be rotated to unwind the system. *The model then transforms like a spinor.*

### 8.1.1 Mach's Principle and Half-Integral spin

There is a very strong physical reason for accepting constraints on the diffeomorphisms given above. It is reasonable to expect the metric at large distances to be fixed by the matter density of the stars *ie.* Mach's principle. There is a fundamental difference between a manifold representing a single compactly supported topological structure and that of the same structure on a manifold with substantial matter in the neighbourhood of infinity. Only in the latter case is it reasonable to restrict attention to diffeomorphisms which are trivial at infinity.

### 8.1.2 Relevance to 4-Geons

The model of a particle described in this thesis has much in common with the classes of manifolds described by Friedman and Sorkin. In particular, the 4-geons were required to be asymptotically flat, the metric at large values of  $r$  is fixed, being dominated by distant matter. Any transformation of a 4-geon must be one that keeps the metric at infinity constant; thus one of the requirements of Sorkin's model is met. The key property required for spin-half is the inability to define a rotational vector field consistently throughout the entire manifold manifold. Naively, a manifold with CTCs cannot have even a well-defined hypersurface, and since a rotational vector field would be defined upon a hypersurface it looks as if one cannot exist.

However, this argument cannot be sound since two such manifolds side by side would clearly still have CTCs and lack a global hypersurface, but if each one had spin-half then the result is known to have spin-one or spin-zero, depending upon the relative alignment. Furthermore, if the simplest 4-geons are spin-half for the reasons given, then by implication the photon must be more complicated, even

a composite particle.

It therefore appears that there are good grounds for relating spin-half to 4-geons, but the simple assumption that spin-half is a *consequence* of the CTCs cannot be sustained; an examination of this relationship will be the subject of later work.

Friedman and Sorkin constructed a spinor field within the framework of quantum gravity; we can simply use a spinor field,  $\Psi$ , to represent the probability of the 4-geon having a certain position  $\Psi(\mathbf{x})$  or momentum  $\Psi(\mathbf{k})$ . This is the same process as used to describe a spinless particle using a scalar wavefunction (and getting Schrödinger's equation) The properties of the particle, and hence  $\Psi$ , under rotations requires  $\Psi$  to be a spinor field - and the Dirac equation is required rather than Schrödinger's equation.



## Chapter 9

# Implications and Conclusions

### 9.1 The Quest for Exact Particle-like Solutions

Finding exact solutions to Einstein's equations is notoriously difficult. Even Einstein himself doubted if exact solutions would ever be found (although within a year Schwarzschild produced his famous spherically symmetric solution).

The 4-geon structures envisaged in this thesis are non-linear, topologically non-trivial, probably lacking in symmetry and certainly having a non-trivial causal structure. Searching for exact solutions with the required properties is likely to be futile. Examining the consequences, particularly the testable consequences of the theory is far more likely to be fruitful. The consequences may either reveal inconsistencies in the theory or give more clues to the nature of an exact solution.

### 9.2 The Co-existence of Classical Objects

The theory shows how certain structures in general relativity will behave according to quantum mechanics, and conversely that the distinguishing features of quantum mechanics can be derived using topological structures in spacetime. There is no reason why objects described *precisely* by classical mechanics cannot exist - such

a classical object would not exhibit wave-particle duality, but would have well-defined (if unknown) values for position and momentum simultaneously. In principle, an evolving three manifold would behave like a classical object, and, conversely, a classical object would not have the complicated causal structure which we are postulating for quantum particles.<sup>1</sup>

That classical and quantum objects can, in principle, exist side by side shows vividly that quantum mechanics is *not* simply about the difficulty of measuring some parameters simultaneously, but is a far deeper fact that the properties of a quantum particle are not even defined until a measurement is made. Even for a classical particle there would be the same impossibility of simultaneously measuring position and momentum; however values for position, momentum, and spin *could* be assigned in a consistent way and the measurements would show this. The Kochen-Specker paradox shows that spin measurements for a classical object cannot have the same statistics as for a quantum mechanical spin-one particle.

### 9.3 Gravitational Waves

Gravitational waves are exact solutions of Einstein's vacuum equations which have well defined properties and lack the complicated causal structure which we have exploited to explain quantum effects. It therefore follows that gravitational waves will not exhibit quantum features. They are classical entities. This is a remarkable conclusion which is an inextricable feature of this theory - as opposed to any theory of quantum gravity. There is no place for a graviton as a quantum of gravitational waves in this theory. Unlike the electromagnetic case, energy can be gained or lost in any quantity and gravitational waves of arbitrary weakness can arrive at a detector. The gravitational equivalent of the photo-electric effect cannot take place; there is no wave particle duality for gravitational waves.

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<sup>1</sup>Brill and Hartle describe such a solution [9] they have defended its validity recently[1].

## 9.4 The Particle Spectrum

The implication of this theory is that the particle spectrum can be obtained by finding solutions to the field equations of general relativity which have the required properties of self-interaction via CTCs. The asymptotic flatness axiom is not an essential feature; for confined objects such as quarks it is probably violated. The property of confinement may even be explained by a lack of asymptotic flatness. The gravitational self-energy could increase with distance around a single quark; such an increase would not only be a source of pair creation, but also be in contradiction to the known flat background metric which we observe and which is fixed by all the matter in the surrounding universe.

## 9.5 Gravitational Collapse

There is no known mechanism for preventing total gravitational collapse of sufficiently massive bodies. In the core of large black holes, matter will continue to collapse inside the event horizon until it becomes infinitely dense and spacetime has infinite curvature at that point. While the singularity is hidden by the event horizon it can have no observable consequences<sup>2</sup>. Yet the occurrence of such singularities has profound significance for the theory, since it shows that Einstein's theory breaks down. The theory of general relativity models spacetime as a semi-Riemannian manifold, which requires it to be non-singular! Infinities in general relativity are as abhorrent as in any other physical theory. The need to avoid singularities is one of the motivating factors behind the quest for quantum gravity; in much the same way that quantum mechanics avoided problems in electromagnetism such as the collapse of an atom and the infinite energy of black-body radiation.

The earliest theories predicting singularities made major assumptions about the symmetry. More recent work has extended the results to cover all real cases and

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<sup>2</sup>The metric outside the event horizon of a blackhole is independent of the diameter of the source



the collapse now looks inevitable.

An analysis of the singularity theorems is beyond the scope of this thesis, but the lack of a global hypersurface certainly invalidates some work (see for example [18, chapter 1]). Whilst the possibility that the metric itself is indeterminate is not considered by any theorems. Our new theory therefore offers, within Einstein's original equations, the possibility of avoiding gravitational collapse.

## 9.6 Reconciliation of General Relativity and Quantum Mechanics

Another justification for a quantum theory of gravitation is to reconcile the classical gravitational field and the quantum characteristics of particles - this we have done. In doing so a great deal that was taken for granted regarding the metric has been lost; the metric is no longer well defined, and time as we know it only exists as an asymptotic property far away from a particle.

Alternative ways of reconciling general relativity and quantum mechanics are no less speculative. Some postulate new untested equations, some assume that spacetime has extra dimensions, relying on an unknown mechanism to cause surplus dimensions to disappear from view. Some such theories continue to be reported, despite predicting a spectrum of unobserved particles; an unknown mechanism is again called upon to create appropriate masses. Some theories rely on all these speculative features and more besides, yet none have the same unifying potential as that presented here. All other theories that reconcile gravitation and quantum mechanics rely on abandoning general relativity - for which there is no experimental justification or motivation.

The great beauty of the present explanation is that it requires no new theory - it simply exploits in a novel way, certain possibilities offered by the existing theory of general relativity.

## 9.7 The Use of CTCs

The most revolutionary aspect of this theory is the use of CTCs as part of the structure of an elementary particle. But far from being an addition to the theory of general relativity, CTCs are a natural consequence of accepting the mathematics of the Einstein's theory. The burden of proof lies with opponents of CTCs to explain why they do not exist in general relativity or to give a new theory which forbids CTCs and which supplants general relativity. Not surprisingly, opinion seems to be moving away from the search for a protection mechanism to the realisation that perhaps CTCs would not be so abhorrent after all, and either do exist or at least could exist [37].

## 9.8 Conclusion

It has been argued that this work is speculative. It is. It rests on the assumption that solutions to Einstein's equations exist with the properties outlined in chapter 6. In return for this degree of speculation the work offers an explanation of quantum phenomena in terms of an established theory (general relativity) for the first time ever. It offers a model for an elementary particle along the lines originally sought by Einstein[13, 33]. By showing that general relativity and quantum mechanics are compatible after all, it removes the need for a new and different theory to reconcile them - a goal sought for several decades. No other theory offers to:

1. Reconcile quantum mechanics and general relativity.
2. Explain the strange properties of quantum mechanics.
3. Explain what a particle is.
4. Explain the origin of charge and mass[31].
5. Unify the particle and field descriptions of Nature.

It is quite amazing for any theory to give a consistent and unified explanation for all these phenomena. What is even more surprising is how few assumptions are required. This is in marked contrast to earlier programmes aimed at unifying general relativity and quantum mechanics; in particular:

1. Einstein's theory of relativity is unmodified.
2. Quantum mechanics is unmodified.
3. The number of space and time dimensions is  $3+1$ .
4. There is no need to postulate the existence of particle fields as fundamental entities.
5. There is no requirement for sources of charge, at a fundamental level.

In short, the level of speculation in this work is substantially less than that in any other theories - yet the degree of unification offered is far greater.

It has always been a feature of great theories that they have applications far beyond the original motivation. Newton's equations are valid for atoms and spacecraft alike - which were unknown in his day. Electromagnetism which explained laboratory experiments is still valid even inside an atom, despite the considerable doubts as the structure of atoms was explored. Now the theory of general relativity has been applied to the quantum arena and, rather than being discarded, as many people expected, it is found to have the power to explain the previously inexplicable quantum phenomena.

It would indeed be ironic if the interpretation of quantum theory with which Einstein was so dissatisfied could be seen to be a consequence of his own General Theory of Relativity!



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